Winner-Take-All Crowdsourcing Contests with Stochastic Production

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Abstract

We study winner-take-all contests for crowdsourcing procurement in a model of costly effort and stochastic production. The principal announces a prize value P, agents simultaneously select a level of costly effort to exert towards production, yielding *stochastic* quality results, and then the agent who produces the highest quality good is paid P by the principal. We derive conditions on the probabilistic mapping from effort to quality under which this contest paradigm yields efficient equilibrium outcomes, and demonstrate that the conditions are satisfied in a range of canonical settings.

Introduction

Crowdsourcing is an increasingly popular model of procurement in today's online marketplaces. A principal seeks completion of a task, posts an open call for submissions, and allows multiple agents (workers) to simultaneously submit solutions, awarding a prize to the participant with the best solution. The number and size of online crowdsourcing marketplaces has grown markedly in recent years; notable examples include Taskcn, Topcoder, 99designs and CrowdFlower. Crowdsourced tasks resemble contests in that many agents simultaneously exert effort in an attempt to win a prize, where the results are determined based on relative performance. The process of selecting a winner based on submission quality and awarding her a lump sum prize constitutes a particular class of contest mechanisms, which we term *winner-take-all*.

In this paper, we initiate a study of winner-take-all contest mechanisms in the model of costly effort and stochastic production introduced in (Cavallo and Jain 2012). In that previous work, we identified a condition under which extreme-effort strategy profiles—i.e., ones where all agents exert either maximum effort or zero effort—are efficient in the sense of maximizing expected value to the principal minus production costs to the workers, and we showed that a number of canonical distributions satisfy this condition. But to implement efficient policies in general, we had to specify a mechanism that departs significantly in form from the contests typical of most crowdsourcing markets.

In the current paper, rather than designing a solution that universally satisfies certain properties, we instead set out to evaluate the winner-take-all schemes that are currently so prevalent in online crowdsourcing. We are motivated by the following questions: What level of effort towards production is induced by winner-take-all contest schemes? How does this depend on the magnitude of the prize on offer? If the principal sets the prize to maximize his value for the produced good minus the prize he must pay, what prize value will he choose? And ultimately, are winner-take-all mechanisms typically efficient?

Since we take agents to be self-interested and strategic, the answer to these questions requires a game-theoretic approach and so we will analyze what happens *in equilibrium*. Our main contribution is to identify a sufficient condition on the stochastic mapping from production-effort to output-quality that yields optimal extreme-effort strategy profiles in equilibrium. We show that a number of canonical distributions satisfy this sufficient condition. This coupled with a result from our previous paper (Cavallo and Jain 2012) establishes that optimal extreme-effort strategy profiles are frequently both efficient and achieved in equilibrium by winner-take-all contest mechanisms.

Related Work

A new line of research providing a theory of crowdsourcing contests has recently emerged (DiPalantino and Vojnovic 2009; Archak and Sundararajan 2009; Chawla *et al.* 2012; Cavallo and Jain 2012). Most of this work focuses on the case where agents have private skill information and choose a privately observed level of effort to expend towards production. DiPalantino and Vojnovic (2009) make the connection to all-pay auctions and model a market with multiple contests, considering the principal's optimization problem in the limit-case as the number of agents and contests goes to infinity. Archak and Sundararajan (2009) and Chawla *et al.* (2012) focus on the design of a single contest, seeking to determine how many prizes should be awarded and of what value. Chawla *et al.* (2012) make the connection between crowdsourcing contests and optimal auction design, finding that the optimal crowdsourcing contest—from the perspective of the principal seeking a maximum quality submission—is a virtual valuation maximizer.

Outside of (Cavallo and Jain 2012), a principal-centric viewpoint has most frequently been adopted. Design-oriented work has focused on how to maximize submission quality given a prize budget, whether it be the highest quality submission (Moldovanu and Sela 2006; Chawla *et al.* 2012) or the total sum of submission qualities (Moldovanu and Sela 2001; 2006; Minor 2011). Other work has focused on maximizing the sum of the top k submissions minus the prize awarded (Archak and Sundararajan 2009). Ghosh and Hummel (2012) consider a more general class of utility functions to optimize for in a setting with virtual "points" (a type of currency); namely, they show that there is a *best contribution mechanism* that can implement the principal-optimal outcome (in their model, the principal does not experience disutility for the prize awarded). Another line of work in the economics literature uses contests to extract effort under a hidden action (Lazear and Rosen 1981; Green and Stokey 1983; Nalebuff and Stiglitz 1983). As in our model, the output is a stochastic function of the unobservable effort, but the setting is different in that the principal obtains value from the cumulative effort of the agents rather than just the maximum result.

These works are related to a long literature in economics focused on providing equilibrium characterizations for all-pay auctions in complete information settings (Tullock 1980; Moulin 1986; Baye *et al.* 1996; Bertoletti 2010) as well as incomplete information settings (Weber 1985; Hilman and Riley 1989; Krishna and Morgan 1997; Amann and Leininger 1996). There has also been work on sequential all-pay auctions (Konrad and Leininger 2007; Segev and Sela 2011; Liu *et al.* 2011) and multi-stage research tournaments that award a single prize (Taylor 1995; Fullerton and McAfee 1999).

Like the current paper, most of this previous work assumes strategic agents and thus consists largely of equilibrium analysis. However, a critical differentiator of the current paper is the production model: we adopt the model of (Cavallo and Jain 2012), where quality is a *stochastic* function of effort and skill. From an efficiency perspective, this stochasticity (combined with a deadline under which procurement is required) is the most natural way we could think of to motivate the crowdsourcing paradigm—the redundant costly-effort of simultaneous production is justified by the principal's value for high quality and inability to ensure receipt before the deadline if production were instead ordered sequentially. In our previous paper we introduced this model and designed efficient mechanisms for the problem of crowdsourcing; here we instead analyze winner-take-all mechanisms, currently the prevailing crowdsourcing payment scheme seen in practice. Our goal in doing so is to determine the effectiveness of winner-take-all contests from an efficiency perspective. We aim to quantify the efficiency gap, or the tradeoff between implementing the complicated yet perfectly efficient mechanisms of (Cavallo and Jain 2012) versus the simpler winner-take-all mechanisms. To the best of our knowledge, no previous work has addressed the efficiency of winner-take-all contests or studied them in a model of stochastic production.[1]

Model

There is a set of agents $I = \{1, \ldots, n\}$ (with $n \geq 2$) capable of producing goods for a principal, where each $i \in I$ makes a *privately observed* choice of effort δ_i to expend on production. We identify effort level δ_i with the dollar value in costs ascribed to it by agent i. We assume that $\delta_i \in [0, 1]$, $\forall i \in I$.[2] If an agent attempts production with effort δ_i, a good is produced with quality that is a priori uncertain but is a function of δ_i.

Quality is identified with value to the principal in dollar-terms, and can be thought of as the output of a non-deterministic function mapping effort to $\Re^+$. The probability distribution over *relative* quality, given any effort level, is publicly known; the principal has value $v \in \Re^+$, a scale factor that maps relative quality levels to absolute quality (dollar-value to the principal).[3] The principal obtains value commensurate with the maximum-quality good produced. For instance, if there are two agents i and j who expend effort δ_i and δ_j and produce goods with relative qualities q_i and q_j, the principal's utility from obtaining the goods will equal $\max\{vq_i, vq_j\}$.

For any non-zero effort level $\delta_i \in (0, 1]$, we denote the p.d.f. and c.d.f. over resulting *relative quality* as f_{δ_i} and F_{δ_i}, respectively, where the support of f_{δ_i} is in $\Re^+$. We assume homogeneity across agents in the sense that the private effort choice is the only differentiating factor; i.e., for two agents making the same choice of effort level, the distribution over the quality they will produce is the same (though there is no presumed correlation so the resulting quality may differ).[4] We assume the probability density over quality, evaluated at any particular quality level, is differentiable with respect to effort δ_i, for all $\delta_i \in (0, 1]$. We assume that effort $\delta_i = 0$ deterministically yields quality 0 (no effort yields no production), and then for notational convenience we let $F_0(x) = 1$, $\forall x \geq 0$. We also assume that, $\forall x > 0$, $F_{\delta_i}(x)$ converges to 1 as δ_i goes to 0.[5] Finally, we make the natural assumption that more effort has first-order stochastic dominance over less effort with respect to quality, i.e.:

$$\forall 0 \leq \delta_i < \delta_i' \leq 1,\ \forall x \in [0, v],\ F_{\delta_i}(x) \geq F_{\delta_i'}(x)$$

[1]While we lack the space to go into great detail here, it is worth noting that in the special case of *deterministic* production functions any equilibrium of the winner-take-all contest will: i) be efficient, and ii) have only a single agent exerting effort towards production. This provides an intuitive motivation for the stochastic production model, since in the deterministic case *crowdsourcing* (i.e., work done by a *crowd* rather than a lone individual) neither occurs in equilibrium nor is efficient.

[2]The only loss in generality here is in assuming a finite bound on effort.

[3]Note that in (Cavallo and Jain 2012) the principal's value v was *private*. Here, because the principal is essentially setting the mechanism by choosing a prize value, whether v is private or public is of no consequence to the equilibrium outcomes that result.

[4]In (Cavallo and Jain 2012), a richer model is also considered where agents are also differentiated by private skill information (see also (Chawla *et al.* 2012)).

[5]This entails that, $\forall \delta_{-i} \in [0, 1]^{n-1}$ such that $\max_{j \in I \setminus \{i\}} \delta_j > 0$, $\int_0^\infty f_{\delta_i}(x) \prod_{j \in I \setminus \{i\}} F_{\delta_j}(x) dx$ converges to 0 as δ_i goes to 0. In words: as long as some agent other than i is exerting non-zero effort, i's probability of winning the contest (and, with it, his expected utility) goes to 0 as his effort goes to 0.

We will analyze a particular family of mechanisms that we call winner-take-all contests. In such contests, a game is defined wherein the principal chooses a prize value P and each agent $i \in I$ subsequently chooses an effort level δ_i, simultaneously with the other agents' effort choices. To simplify things, we will assume that each agent's strategy space consists only of *pure* strategies, i.e., any deterministic choice of effort level in $[0, 1]$. We adopt a quasilinear utility model and assume all players are risk-neutral. Given our identification of the quality of the good (scaled by the principal's private value v) with the dollar value ascribed to it by the principal, a rational principal will choose a prize P that induces an effort profile which maximizes his expected value for the highest quality good produced minus P. Likewise, given our identification of effort level δ_i with the dollar value in costs ascribed to it by agent i, in equilibrium each agent will choose δ_i to maximize P times his probability of producing the highest-quality good—given the other agents' effort choices—minus δ_i.

Letting $Q_j(v, \delta_j)$ be a random variable representing the absolute quality level produced by agent j when he expends effort δ_j, given the principal's value v, the expected efficiency of an equilibrium in which each $i \in I$ exerts effort δ_i is:

$$\mathbb{E}[\max_{i \in I} Q_i(v, \delta_i)] - \sum_{i \in I} \delta_i$$

(The prize value P does not factor into social welfare since it is subtracted from the principal's utility and added to the winning agent's.) An *efficient* effort policy is a vector of effort levels that maximizes the above equation given v.[6]

In the next section, we present our main results, establishing several facts about equilibrium agent-level responses to any given prize choice by the principal, providing conditions under which extreme-effort strategies will be equilibria, and then showing that in such cases the principal has an equilibrium choice of prize value P that leads to maximum efficiency. We then show that these results entail the existence of efficient equilibria in many canonical cases. However, the results do not preclude the existence of alternative inefficient equilibria as well, so we also provide a full equilibrium characterization for one case, that where quality is distributed uniformly over a range that increases linearly with effort.

General Equilibrium Properties

Our first set of results will consider agent-level strategies that are equilibrium responses to a given choice of prize value by the principal. We define:

Definition 1 (agent-level Nash equilibrium). *Given a choice of prize value P, a set of effort levels δ for the agents constitutes an agent-level Nash equilibrium if and only if, $\forall i \in I$, δ_i is a best-response to P and the other agents' chosen effort levels δ_{-i}.*

We start by observing that no equilibrium will ever involve cumulative agent effort exceeding the prize value, and also no equilibrium will involve only a single agent exerting non-zero effort. These facts will be useful in deriving the full equilibrium results to come.

Proposition 1. *For arbitrary quality distributions and arbitrary prize value $P \in \Re^+$, any agent-level Nash equilibrium strategy profile has agents collectively exerting at most P units of effort.*

Proof. Consider arbitrary prize $P \in \Re^+$ and effort profile $\delta_1, \ldots, \delta_n$. Assume δ is an agent-level Nash equilibrium given P. Each agent's expected utility is her expected prize reward (denoted R_i for the purposes of this proof) minus effort exerted, i.e., $\mathbb{E}[R_i(P, \delta_1, \ldots, \delta_n)] - \delta_i$. Each agent must have an expected utility of at least 0 since otherwise an agent could beneficially deviate by exerting 0 effort and receive a reward of 0. Therefore, $\sum_{i=1}^{n} (\mathbb{E}[R_i(P, \delta_1, \ldots, \delta_n)] - \delta_i) \geq 0$. But $\sum_{i=1}^{n} \mathbb{E}[R_i(P, \delta_1, \ldots, \delta_n)] = P$, and therefore $\sum_{i=1}^{n} \delta_i \leq P$. □

Proposition 2. *For arbitrary quality distributions and arbitrary prize value $P \in \Re^+$, no agent-level Nash equilibrium has exactly one agent exerting non-zero effort.*

Proof. Consider a candidate equilibrium profile with one agent exerting x units of effort, where $0 < x \leq 1$, and others exerting 0. This agent's expected utility is $P - x$. If the agent reduces his effort to $\frac{x}{2}$, his expected payoff increases to $P - \frac{x}{2}$, a profitable deviation. □

Proposition 2 tells us that when an efficient effort profile involves exactly one agent exerting full effort, the winner-take-all contest cannot achieve it in equilibrium. In (Cavallo and Jain 2012, Lemma 1), we gave a sufficient condition for efficiency of extreme-effort strategy profiles, which we reproduce in a slightly modified form here:

Condition 1 (Efficiency of extreme-effort). $\forall i \in I$, $\forall \delta_{-i} \in [0, 1]^{n-1}$, $\forall v \in \Re^+$, $\forall a, b \in \Re^+$ *with* $a < b$, $\forall \beta \in \Re^+$, $\forall \epsilon \in [a, b)$,

$$-\frac{\partial}{\partial \delta_i}\Big(v \int_\beta F_{\delta_i}(x)\, dx\Big)\Big|_{\delta_i = \epsilon} > 1 \tag{1}$$

$$\Rightarrow\ -\frac{\partial}{\partial \delta_i}\Big(v \int_\beta F_{\delta_i}(x)\, dx\Big)\Big|_{\delta_i = k} > 1,\ \forall k \in [\epsilon, b] \tag{2}$$

While the condition is somewhat opaque, it allows for straightforward demonstration that the set of efficient effort policies for a distribution consists *only* of extreme-effort policies (this follows almost immediately from the proof of Lemma 1 of (Cavallo and Jain 2012)). Moreover, when the condition holds, there always exists a value v for the principal such that the only efficient policy has exactly one agent participate. This in combination with Proposition 2 gives us the following:

Proposition 3. *For any distribution that satisfies Condition 1, there exists a value v for the principal such that the winner-take-all contest has no efficient equilibria.*

We now derive conditions under which extreme-effort strategy profiles are an agent-level Nash equilibrium.

[6] That is, effort vector δ^* is efficient if and only if $\delta^* \in \arg\max_{\delta \in [0,1]^n} (\mathbb{E}[\max_{i \in I} Q_i(v, \delta_i)] - \sum_{i \in I} \delta_i)$.

Lemma 1. *For arbitrary $m \in \{2, \dots, n-1\}$, m agents exerting effort 1 and $n-m$ agents exerting effort 0 is an agent-level Nash equilibrium if and only if the prize value P satisfies: $m \le P \le m+1$ and, $\forall \delta \in (0,1)$,*

$$\frac{1-\delta}{\frac{1}{m} - \int f_\delta(x) F_1(x)^{m-1}\, dx} \le P \le \frac{\delta}{\int f_\delta(x) F_1(x)^m\, dx} \tag{3}$$

All n agents exerting effort 1 is an agent-level Nash equilibrium if and only if, $\forall \delta \in [0,1)$,

$$P \ge \frac{1-\delta}{\frac{1}{n} - \int f_\delta(x) F_1(x)^{n-1}\, dx} \tag{4}$$

Proof. Consider arbitrary $P \in \Re^+$ and $m \in \{1, \dots, n\}$. m agents exerting effort 1 and $n-m$ agents exerting effort 0 is an agent-level Nash equilibrium if and only if, $\forall \delta \in [0,1]$,

$$P\left[\int f_1(x) F_1(x)^{m-1}\, dx - \int f_\delta(x) F_1(x)^{m-1}\, dx\right] \ge 1-\delta, \tag{5}$$

$$\text{and} \quad P \int f_\delta(x) F_1(x)^m\, dx \le \delta \tag{6}$$

(with Eq. (6) voided in the case of $m = n$).

The left-hand-side of Eq. (5) is the expected utility loss (excluding costs of effort) to an agent from moving from effort level 1 to δ, when $m-1$ other agents are exerting effort 1 (and all others are exerting 0). In a Nash equilibrium this will (weakly) exceed the cost "savings" (i.e., the right-hand-side). Similarly, the left-hand-side of Eq. (6) is the expected utility gain (excluding costs of effort) to an agent from moving from effort level 0 to δ when m other agents are exerting effort 1 (and all others are exerting 0). In a Nash equilibrium this will not exceed the added cost (i.e., δ).

Noting that $\int f_1(x) F_1(x)^{m-1}\, dx = 1/m$, $\forall m \in \{2, \dots, n-1\}$: for $\delta = 0$ Eq. (5) entails $P \ge m$, for $\delta = 1$ Eq. (6) entails $P \le m+1$, and $\forall \delta \in (0,1)$, Eqs. (5) and (6) can be combined and rewritten as Eq. (3). Similarly, in the case of $m = n$ Eq. (5) holds trivially for $\delta = 1$ and for all other values of δ can be rewritten as Eq. (4). □

Proposition 4. *If prize $P \in (0,2)$, there is never an extreme-effort agent-level Nash equilibrium. For arbitrary prize $P \in [2,n]$, $\forall m \in \{2, \dots, n\}$, if m agents exerting effort 1 and $n-m$ exerting effort 0 is an agent-level Nash equilibrium, then $m = \lfloor P \rfloor$ or $P \in \{3, \dots, n\}$ and $m = P-1$.*

Proof. Consider arbitrary prize $P \in (0,2)$. By Proposition 1, if there is an extreme-effort agent-level Nash equilibrium it must have $m < 2$ agents participating. But $m = 1$ cannot be an equilibrium by Proposition 2, and $m = 0$ cannot be an equilibrium when there is a positive prize on offer. The rest of the proposition follows directly from Lemma 1. □

Theorem 1. *For arbitrary prize value $P \in [2,n]$, letting $m = \lfloor P \rfloor$, the following is a* ***sufficient*** *condition for m agents exerting effort 1 and $n-m$ exerting 0 as an agent-level Nash equilibrium: $\forall \delta \in (0,1)$,*

$$\delta \ge \max\left\{m \int f_\delta(x) F_1(x)^{m-1}\, dx, \right. \tag{7}$$
$$\left. (m+1) \int f_\delta(x) F_1(x)^m\, dx\right\}$$

Proof. Consider arbitrary prize $P \in [2,n]$ and let $m = \lfloor P \rfloor$. Then from Lemma 1 (rearranging Eq. (3)), necessary and sufficient conditions for m being an extreme-effort agent-level Nash equilibrium are: $\forall \delta \in (0,1)$,

$$\delta \ge 1 - P\left[\frac{1}{m} - \int f_\delta(x) F_1(x)^{m-1}\, dx\right], \text{ and} \tag{8}$$

$$\delta \ge P \int f_\delta(x) F_1(x)^m\, dx, \tag{9}$$

where Eq. (9) is not required if $P = n$. Since $P \ge m$, a stronger version of Eq. (8) is $\delta \ge 1 - m[\frac{1}{m} - \int f_\delta(x) F_1(x)^{m-1}\, dx] = m \int f_\delta(x) F_1(x)^{m-1}\, dx$. Similarly, since $P < m+1$, a stronger version of Eq. (9) is $\delta \ge (m+1) \int f_\delta(x) F_1(x)^m\, dx$. Thus the two inequalities will be satisfied if Eq. (7) holds. □

Theorem 2. *For arbitrary prize value $P \in [2,n)$, letting $m = \lfloor P \rfloor$, the following is a* ***necessary*** *condition for m agents exerting effort 1 and $n-m$ exerting 0 as an agent-level Nash equilibrium: $\forall \delta \in (0,1)$,*

$$\delta \ge \max\left\{(m+1) \int f_\delta(x) F_1(x)^{m-1} dx - \frac{1}{m}, \right. \tag{10}$$
$$\left. m \int f_\delta(x) F_1(x)^m dx\right\}$$

Proof. Consider arbitrary prize $P \in [2,n)$ and let $m = \lfloor P \rfloor$. Assume for contradiction that m is an extreme-effort agent-level Nash equilibrium and, for some $\delta \in (0,1)$, $\delta < \max\{(m+1) \int f_\delta(x) F_1(x)^{m-1} dx - 1/m,\ m \int f_\delta(x) F_1(x)^m dx\}$. As shorthand, let λ denote $1/m - \int f_\delta(x) F_1(x)^{m-1} dx$. Since $P \le m+1$, if $\delta < (m+1) \int f_\delta(x) F_1(x)^{m-1} dx - 1/m$, then:

$$\begin{aligned}
\delta &< (m+1)\Big(\frac{1}{m} - \lambda\Big) - \frac{1}{m} \\
&= \frac{1}{m}(m+1-1) - (m+1)\lambda \\
&= 1 - (m+1)\lambda \\
&\le 1 - P\lambda \\
&= 1 - P\left[\frac{1}{m} - \int f_\delta(x) F_1(x)^{m-1} dx\right]
\end{aligned}$$

This contradicts Eq. (8), which is entailed by the Nash equilibrium conditions of Eq. (3). Now alternatively assume that for some $\delta \in (0,1)$, $\delta < m \int f_\delta(x) F_1(x)^m dx$. Since $P \ge m$, this implies that:

$$\delta < P \int f_\delta(x) F_1(x)^m dx$$

This contradicts Eq. (9), which is also entailed by the Nash equilibrium conditions of Eq. (3). □

The above gives us a handle on agent equilibrium responses to any given prize choice. In order to have a complete equilibrium description, we will now incorporate the principal's strategic choice of prize value. The setting can be viewed as a two-stage game: in stage 1 the principal sets the prize value, and in stage 2 the agents exert effort towards production. Thus we will be concerned with strategy profiles that constitute *subgame perfect equilibria*, defined as follows: (P, δ) is a subgame perfect equilibrium if and only if δ is an agent-level Nash equilibrium given P, and there exists no alternate prize choice that yields (only) agent-level Nash equilibria such that the principal's expected utility is improved over the (P, δ') case, for every δ' that is an agent-level Nash equilibrium given P.

Lemma 2. *For arbitrary quality distribution and arbitrary principal value $v \in \Re^+$, if prize value $P \in \Re^+$ yields an efficient effort profile δ^* as an agent-level Nash equilibrium, with $\sum_{i\in I} \delta_i^* = P$, then (P, δ^*) is a subgame perfect Nash equilibrium.*

Proof. Consider arbitrary quality distribution, arbitrary principal value $v \in \Re^+$, and arbitrary prize $P \in \Re^+$ that yields an efficient effort profile δ^* as an agent-level Nash equilibrium, with $\sum_{i\in I} \delta_i^* = P$. To prove the lemma it is sufficient to show that there exists no P' that induces an agent-level Nash equilibrium δ' with $\mathbb{E}[Q(v, \delta')] - P' > \mathbb{E}[Q(v, \delta^*)] - P$. Assume for contradiction that there does exist such a P' inducing such a δ'. Then:

$$\begin{aligned}
\mathbb{E}[Q(v, \delta')] - P' &\leq \mathbb{E}[Q(v, \delta')] - \sum_{i\in I} \delta_i' \\
&\leq \mathbb{E}[Q(v, \delta^*)] - \sum_{i\in I} \delta_i^* \\
&= \mathbb{E}[Q(v, \delta^*)] - P
\end{aligned}$$

The first inequality holds since $\sum_{i\in I} \delta_i' \leq P'$, by Proposition 1; the second holds by efficiency of δ^*. We've reached a contradiction, and the lemma follows. □

We can now establish the following somewhat surprising result: when extreme-effort is efficient and the conditions for extreme-effort as an agent-level Nash equilibrium obtain (Theorem 1), the winner-take-all contest will yield an efficient strategy profile in subgame perfect equilibrium.

Theorem 3. *For arbitrary quality distribution f_δ and principal value $v \in \Re^+$, if an extreme-effort policy with $m \in \{2, \ldots, n\}$ participants is efficient and Eq. (7) is satisfied for m and f_δ, then this efficient policy is yielded in a subgame perfect Nash equilibrium.*

Proof. Consider arbitrary principal value $v \in \Re^+$ and arbitrary quality distribution f_δ such that an extreme-effort policy with $m \in \{2, \ldots, n\}$ participants is efficient and Eq. (7) is satisfied for m and f_δ. By Lemma 2 it is sufficient to show that if the principal chooses prize value $P = m$, an extreme-effort strategy profile with m participants is an agent-level Nash equilibrium. But this follows immediately from Theorem 1, given that Eq. (7) is satisfied. □

This is good news for proponents of winner-take-all contests. However, we should take care to note that there is a significant distance between an efficient strategy profile being part of *a* subgame perfect Nash equilibrium and it being *the* outcome that results. We have not ruled out the possibility of inefficient agent-level Nash equilibria, for any chosen prize P. When a particular prize value leads to a multiplicity of agent-level Nash equilibria, there is ambiguity about which (if any) should be more "expected"—by a social planner analyzing the game, but also by the principal seeking to optimize his prize choice.

However, because extreme-effort is simple (one either exerts all effort or stays home), one could argue that it is also *focal*, more likely than others to occur in practice; still, the results admit at least the possibility of a second extreme-effort equilibrium with one less agent producing than would be optimal (this possibility is borne out in the case of uniformly distributed quality, as we will see). Given these considerations, we cannot conclude from Theorem 3 that efficiency is the *only* plausible outcome given rational agents and a rational principal, but one might say the efficient outcome is at least supported as a focal equilibrium.

Some canonical distributions

The results of the previous section, culminating in Theorem 3, point a way towards establishing that a winner-take-all contest will have an efficient equilibrium for a given setting. The "setting" is defined by the quality distribution, the number of agents, and the principal's value. By Theorem 3, it is sufficient to demonstrate that a given quality distribution satisfies Condition 1 and Eq. (7) for the efficient number of participants m. Of greatest interest is whether these conditions hold for quality distributions that we think are representative of the real world. But since at this time we have no data on "real world" quality distributions, we will instead examine a diverse set of canonical distributions and show that the conditions hold. In some cases we are able to do this analytically (e.g., for the uniform distribution, which we address in the next section); in others, to establish existence of efficient equilibria we rely on numerical computations.

To start, let us consider the case of the truncated normal distribution over $[0, 1]$ with location parameter $\mu = \delta$ and scale parameter $\sigma = 0.15$. For $m = 3$, the necessary conditions for an extreme-effort equilibrium (Theorem 2) are satisfied, but the sufficient conditions (Theorem 1) are not, leaving existence ambiguous. However, for all larger values of m the sufficient conditions are also satisfied, establishing that there is an efficient equilibrium when the principal's value exceeds 4. The sufficient conditions of Theorem 1 can be restated as saying, $\forall \delta \in [0, 1]$,

$$m \int f_\delta(x) F_1(x)^{m-1}\, dx - \delta \leq 0, \text{ and} \tag{11}$$

$$(m+1) \int f_\delta(x) F_1(x)^{m}\, dx - \delta \leq 0 \tag{12}$$

Figure 1 illustrates the conditions for $m = 4$, demonstrating that they are in fact satisfied. We plot the left-hand-sides of Eq. (11) and Eq. (12) as a function of δ, and note that neither is greater than 0 at any point.

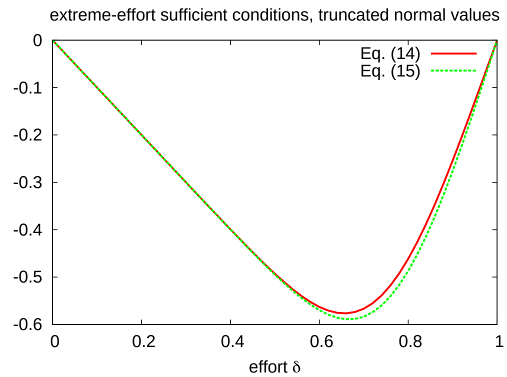

Figure 1: Sufficient conditions for existence of an extreme-effort equilibrium are satisfied for the truncated normal distribution with $\mu = \delta$, $\sigma = 0.15$, and $m = 4$.

In general, we empirically find that across an array of distributions there is a threshold value for m, below which the sufficient conditions for an extreme-effort equilibrium are not satisfied, and above which they are. As described above, for the truncated normal distribution with $\mu = \delta$ and $\sigma = 0.15$, that threshold is $m = 4$. In the case of the exponential distribution with mean $\frac{c}{\delta}$ for $c > 0$, we find that the sufficient conditions are satisfied for all $m > 4$ (we empirically checked only m between 2 and 100), for all examined values of c (we checked c in the range 0.1 to 10, at increasing increments of 0.1).

In the case of the truncated normal distribution with $\mu = \delta$, the threshold level is sensitive to the scale parameter σ. Now varying the scale parameter, again we checked values of m between 2 and 100, and found that all m above a threshold satisfy the sufficiency conditions, but only if σ is low enough. For $\sigma < \frac{1}{3}$ such a threshold exists, which we plot as a function of σ in Figure 2. For $\sigma > \frac{1}{3}$, for values of m even in the thousands there exist no extreme-effort equilibria.

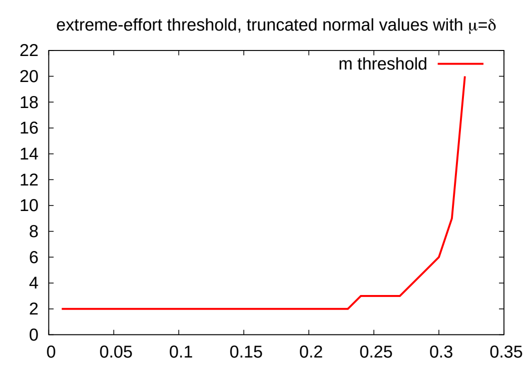

Figure 2: m threshold for truncated normal distribution with mean quality $\mu = 0.5$ as a function of scale parameter σ. For m above this threshold, m as an extreme-effort agent-level Nash equilibrium is verified given $P \in [m, m+1]$.

We also examined quality produced according to a log-normal distribution with various log-scale and shape parameters. Figure 3 depicts quality distributions for the case of log-scale parameter $\mu = -2 + 4\delta$ and shape parameter $\sigma = 0.5$. The threshold for satisfaction of the extreme-effort sufficiency conditions here is $m = 2$, meaning that if there are at least two agents and the principal's value is high enough to warrant at least two participants, then extreme-effort occurs in equilibrium. Since, again, Condition 1 is satisfied for this distribution, Theorem 3 entails that winner-take-all contests have efficient equilibria.

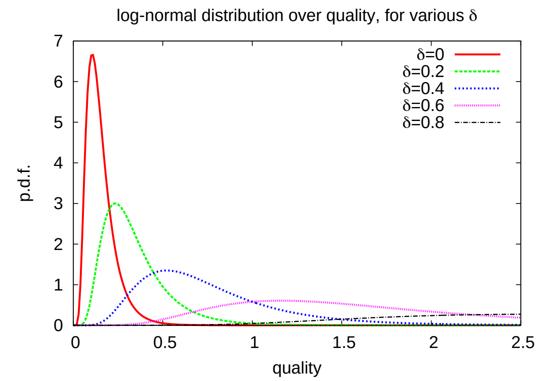

Figure 3: P.d.f. for quality produced according to a log-normal distribution, for various effort levels. The distributions have infinite tails, bounded only below at 0, although the figure only illustrates densities for quality up to 2.5.

Uniformly distributed quality

In this section we characterize the set of equilibria in the case of uniformly distributed quality, a case that is amenable to a complete analytical treatment. Specifically, we assume quality is uniformly distributed between 0 and effort level δ_i; i.e., $\forall \delta_i \in (0, 1]$, $f_{\delta_i}(x) = \frac{1}{\delta_i}$ when $0 \le x \le \delta_i$ and 0 when $x > \delta_i$, and $F_{\delta_i}(x) = \frac{x}{\delta_i}$ when $0 \le x \le \delta_i$ and 1 when $x > \delta_i$. The expected quality yielded by an agent that exerts effort δ_i will equal $\frac{\delta_i}{2}$. The methodology developed earlier in the paper can be very easily deployed to show that efficient equilibria exist here whenever the principal's value exceeds a certain minimum value.

Lemma 3. *The uniform quality distribution satisfies the condition given in Eq. (7) for all $m \in \{2, \ldots, n\}$.*

Proof. For the uniform distribution, for arbitrary $\delta \in (0, 1]$ and $m \in \{2, \ldots, n\}$, $m \int f_\delta(x) F_1(x)^{m-1}\, dx = m \int \frac{1}{\delta} x^{m-1}\, dx = \delta^{m-1}$ and $(m+1) \int f_\delta(x) F_1(x)^m\, dx = \delta^m$. Thus Eq. (7) reduces to: $\forall \delta \in (0, 1)$, $\delta \ge \delta^{m-1}$, which holds for all $m \ge 2$. $\square$

It is easy to verify that for any value $v \ge 6$ for the principal, an efficient effort policy will be extreme-effort with more than 1 agent participating. This combined with Lemma 3 and Theorem 3 immediately yields the following theorem.

Theorem 4. *In the uniformly distributed quality case, if $v \ge 6$ then the efficient effort policy is yielded in a subgame perfect Nash equilibrium.*

Equilibrium characterization

Theorem 4 tells us that extreme-effort strategy profiles are efficient and achieved in equilibrium. However, we will now show something stronger than this, namely, that extreme-effort strategy profiles are the *only* agent-level equilibria (recall our restriction to pure strategies). We will accomplish this with an enumeration of the equilibria over a series of cases covering all possible scenarios.

Lemma 4. *In the uniformly distributed quality case, for arbitrary prize $P \in \Re^+$, any agent-level Nash equilibrium strategy profile has all agents who exert non-zero effort exerting the same amount of effort.*

Proof. Consider arbitrary prize value P and an arbitrary agent-level Nash equilibrium $\delta = (\delta_1, \ldots, \delta_n)$ that consists of i different effort levels $\gamma_1 > \gamma_2 > \ldots > \gamma_i$, with $a_1, \ldots, a_i$ number of agents exerting these efforts, respectively. The expected utility to an agent who exerts effort γ_i can be expressed as: $\frac{\gamma_i^n}{\Pi_{j=1}^i \gamma_j^{a_j}} \cdot \frac{P}{n} - \gamma_i$, i.e., $\gamma_i \cdot \big(\frac{\gamma_i^{n-1}}{\Pi_{j=1}^i \gamma_j^{a_j}} \cdot \frac{P}{n} - 1\big)$. This utility must be at least 0 for each i. If $i > 1$ and an agent exerting γ_i in strategy profile δ instead exerts effort γ_{i-1}, her expected utility will be at least: $\frac{\gamma_{i-1}^n}{(\Pi_{j=1}^{i-2} \gamma_j^{a_j})\gamma_{i-1}^{a_{i-1}+a_i}} \cdot \frac{P}{n} - \gamma_{i-1}$, i.e., $\gamma_{i-1} \cdot \big(\frac{\gamma_{i-1}^{n-1}}{(\Pi_{j=1}^{i-2} \gamma_j^{a_j})\gamma_{i-1}^{a_{i-1}+a_i}} \cdot \frac{P}{n} - 1\big)$. Since $\gamma_{i-1} > \gamma_i$ and $\frac{\gamma_{i-1}^n}{(\Pi_{j=1}^{i-2} \gamma_j^{a_j})\gamma_{i-1}^{a_{i-1}+a_i}} > \frac{\gamma_i^n}{\Pi_{j=1}^i \gamma_j^{a_j}}$, an agent exerting γ_i units of effort has a profitable deviation. This proves that $i = 1$. $\square$

The above lemma radically prunes the space of strategy profiles that are potential agent-level Nash equilibria and is instrumental in establishing the following trio of lemmas that lead to Theorem 5. The proofs are fairly involved and are omitted due to space constraints.

Lemma 5. *In the uniformly distributed quality case, if prize $P > n$, the only agent-level Nash equilibrium has each agent exert full effort.*

Lemma 6. *In the uniformly distributed quality case, if prize $P \in [0, 2]$, all agent-level Nash equilibria have two players exert effort $\frac{P}{2}$ and the remaining agents exert 0.*

Lemma 7. *For any prize $P \in [2, n]$, any effort profile in which $\lfloor P \rfloor$ agents exert full effort and $n - \lfloor P \rfloor$ agents exert 0 is an agent-level Nash equilibrium; $\forall P \in \{3, \ldots, n\}$, any effort profile in which $P - 1$ agents exert full effort and $n - P + 1$ agents exert 0 is also an agent-level Nash equilibrium. This is an exhaustive characterization of the agent-level Nash equilibria for $P \in [2, n]$.*

Lemmas 5, 6 and 7 together give us a complete agent-level Nash equilibrium characterization for winner-take-all contests, for general n:

Theorem 5. *In the case of uniformly distributed quality with $n \geq 2$ agents:*

- *If prize $P \leq 2$, the only agent-level Nash equilibria are those in which exactly two agents exert effort $\frac{P}{2}$ and all others exert 0.*
- *If P is a non-integer in $(2, n)$, then the only agent-level Nash equilibria are those in which exactly $\lfloor P \rfloor$ agents exert full effort and all others exert 0.*
- *If $P \in \{3, \ldots, n\}$, there are two classes of agent-level Nash equilibria: one in which $P - 1$ agents exert full effort and all others exert 0 effort, and the other in which P agents exert full effort and all others exert 0.*
- *If $P > n$, in the unique agent-level Nash equilibrium all n agents exert full effort.*

Extreme-effort profiles are the *only* agent-level Nash equilibria for $P \geq 2$. Combined with Theorem 3, from this we can conclude that in a subgame perfect equilibrium the principal sets P identical to the optimal number of extreme-effort workers m^*, as long as $m^* \geq 2$, and m^* agents will produce (if $m^* > 2$ then there is also an inefficient subgame perfect equilibrium with $P = m^*$ and $m^* - 1$ agents producing). We now complete the picture by examining the case where v is not large enough to warrant at least 2 agents participating in the efficient policy (specifically, when $v < 6$), in which case $P < 2$ may be part of an equilibrium.

Lemma 8. *In the uniformly distributed quality case:*

- *When the principal's value $v < 3$, in the only subgame perfect equilibrium, $P = 0$ and no agents produce.*
- *When $v = 3$, the subgame perfect equilibria are characterized by $P \in [0, 2]$ and two agents exerting effort $P/2$ (with all others exerting 0).*
- *When $v \in (3, 6)$, the subgame perfect equilibria are characterized by $P = 2$ and two agents exerting full effort (with all others exerting 0).*

Proof. From Theorem 4, we know at $v = 6$ the principal sets a prize of $P = 2$ to have two agents exert full effort. Thus for $v < 6$, the principal will set P to be at most 2. From Lemma 6, for $P \in [0, 2]$, two agents will exert effort $\frac{P}{2}$, the others will exert effort 0, and the expected value to the principal of the highest quality good will be $\frac{vP}{3}$. Therefore, the principal's utility is: $\frac{vP}{3} - P = P(\frac{v}{3} - 1)$ for all $P \in [0, 2]$. If $v > 3$, the principal's utility is always positive and maximized when $P = 2$. When $v = 3$, the principal's utility is 0 for all $P \in [0, 2]$. When $v < 3$, the principal's utility is negative for all $P > 0$. $\square$

The efficient policy has no production when $v \in [0, 2]$, one extreme-effort producer when $v \in (2, 6)$, two extreme-effort producers when $v \in [6, 12)$, and more than two extreme-effort producers whenever $v \geq 12$. This combined with the preceding analysis establishes the following characterization of circumstances under which winner-take-all contests are efficient given uniformly distributed quality.

Theorem 6. *In the uniformly distributed quality case, for all $v \in [2, 6)$ the winner-take-all contest has no efficient subgame perfect equilibria; for all $v \in [6, 12)$ the only subgame perfect equilibria are efficient; for all $v \geq 12$, there are two classes of subgame perfect equilibria, one of which is efficient.*

Conclusion

In this paper, we provided a thorough analysis of winner-take-all contest mechanisms in a model of stochastic production, giving conditions under which efficient effort profiles are yielded in equilibrium. From our results, we are able to conclude that for a large range of values held by the principal and for many canonical distributions, winner-take-all crowdsourcing contests have efficient equilibria (though they may not be unique). We also established that for any quality distribution for which extreme-effort strategy profiles are uniquely efficient, there exists a principal's value v for which the winner-take-all crowdsourcing contest has no efficient equilibria.

We should be mindful of the fact that even in the cases where the winner-take-all paradigm yields efficiency in equilibrium, without a centralization procedure it is highly questionable whether the efficient equilibrium will result, as it would require significant agent coordination. For instance, when the set of equilibria has 2 out of the n agents participating with full effort, and the other $n-2$ not participating, how would each agent determine whether or not to participate? Thus a cautious summary of our main results is this: when the principal's value is large enough and agents are rational, a *centrally coordinated* implementation of a winner-take-all contest will frequently be efficient.

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