

# Designing Incentives for Online Question-and-Answer Forums

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## Abstract

In this paper, we provide a game-theoretic model of a problem of sequential information aggregation motivated by online question-and-answer forums. In the model, an asker posts a question and each user decides when to aggregate a unique piece of information with existing reports, in order to form a new answer. At some point the quality of the answer is satisfactory and the asker closes the question and then follows a rule in deciding how to allocate points to users. We characterize the subgame perfect equilibrium under complements and substitutes valuations for information for different scoring rules. A best-answer scoring rule, adopted to model Yahoo! Answers, is effective in promoting an efficient equilibrium for substitutes information, where it isolates an equilibrium in which all users respond in the first round. But we find that the best-answer rule is ineffective for complements information, isolating instead an equilibrium in which all users respond in the final round. In addressing this, we demonstrate that an approval rule and a proportional-share scoring rule enable the most efficient equilibrium for complements information, under certain conditions, by providing incentives for early responders as well as users who contribute more marginal value by waiting, while retaining the efficient equilibrium for substitutes information. We also introduce axioms of *anonymity*, *monotonicity* and *time-invariance*, and establish that no rule can isolate the efficient equilibrium for all substitutes and complements valuations.

## 1 Introduction

This work provides a game-theoretic model of a problem of sequential information acquisition that is motivated by online question and answer forums such as Yahoo! Answers. Yahoo! Answers is a diverse question and answer forum where users can post questions or answer questions on wide variety of topics. Yahoo! Answers has 25 categories ranging from Computers & Internet to Travel to Family & Relationships to Health. Users may post discussion questions, factual questions or polls. In Yahoo! Answers, people do not exchange money for the exchange of information, but instead receive points for contributions that influence leaderboard and top-contributor designations while also allowing users to post their own questions.<sup>1</sup>

We focus on modeling the dynamics in responding to a single question and consider factual questions, such as “What are the main causes of the current housing crisis?”, rather than discussion questions (e.g. “What

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<sup>1</sup>A user is placed into one of seven levels based on her number of points. The higher the level, the greater privileges a user will get in terms of the number of questions she can ask per day. All users have a profile where the number of points the user has, the level, and the percentage of best answers is clearly displayed. In addition to the point system, the “top contributor” is displayed at the top of the page for each semantic category and sub-category and there is a leaderboard of the top ten users.

is your favorite movie of all time?”), so that it makes sense to consider the value of each answer improving over time as users aggregate privately held information with earlier reports.<sup>2</sup>

Whereas the established model of contest design[8, 9] considers agents with costly but independent effort and seeks to maximize the total effort exerted across all agents, in our model each agent can build on the work already contributed by other agents and submit a solution that dominates all solutions so far submitted. Moreover, in keeping the model simple we assume that an agent can contribute an answer for no cost and that the asker is modeled as satisficing, with a private quality threshold at which she will close the question. This leads to an interesting strategic tradeoff. By delaying a contribution, a user runs the risk that the asker will be satisfied with the current answer and close the question. On the other hand, by delaying a user can take advantage of contributions by other users and submit a better answer, thereby increasing the probability that a user’s answer will cross the quality threshold of the asker.

Our interest is in characterizing the structure of the equilibria when users seek to maximize the expected number of points assigned to them by the asker under different scoring rules, and to identify scoring rules that promote equilibria in which all users contribute information in the first round. In the model that we propose, each user has a unique piece of information that is relevant to a question, and can decide when to report this information and aggregate it with previous reports. Thus, as information is reported it is aggregated into the responses, so that the value to the asker monotonically improves while the question remains open. In the case that multiple pieces of information are simultaneously revealed we assume that the asker is able to aggregate the information, and associate each user that contributed in that round with an answer equal in quality to that achieved by aggregating all the information up to and including the current round.

In considering different structure on interactions between individual pieces of information, we consider two distinct cases: a *complements case* in which each successive piece of information is worth more to the asker than the previous one, and a *substitutes case* in which each successive piece of information is worth less than the previous one. The asker privately draws a random value threshold, and is satisfied with any answer with value above this threshold. The asker prefers to receive a satisfactory answer sooner rather than later, and closes the question as soon as the threshold is exceeded.

We first analyze the equilibrium for a best-answer scoring rule, that models the current Yahoo! Answers environment. We find that this scoring rule is effective in isolating a subgame perfect Nash equilibrium in which all users reveal information in the first round for substitutes information. This is the efficient outcome because the asker receives a satisfactory answer as soon as possible for all possible quality thresholds. On the other hand, the best-answer rule is ineffective for complements information, where it in fact isolates the least efficient equilibrium, in which every user posts information in the very last round. The problem is that, for the case of complements information, the relative gain of playing later and combining an answer with previous answers is greater than the relative gain of playing early, to ensure that the answer is in before the question closes.

In addressing this problem, we consider two alternative scoring rules. The first is an *approval rule*, parameterized by integer  $k > 1$ , in which the asker assigns one point to the most recent  $k > 1$  answers (or some random  $k$  subset if more than  $k$  answers were received in the most recent round) upon closing the question. The approval rule enables the most efficient outcome in equilibrium for complements information, under certain restrictions on the valuation function. This is in contrast to best-answer, which is unable to implement the efficient outcome for any complements valuation function. The approval rule also retains the efficient equilibrium in the case of substitutes information. However, the downside of the approval rule is that it also retains the least efficient equilibrium for complements information and introduces the least efficient equilibrium for substitutes information, with certain restrictions on the valuation function. In comparison, the best-answer rule isolates the efficient equilibrium for all substitutes valuation functions. An interesting feature of the approval rule is the tunable parameter  $k$ , which represents the tradeoff between the benefit of this scoring rule for the case of complements information and the disadvantage for the case of substitutes information.

In the *proportional-share* rule, the asker assigns a share of the total available points in proportion to the marginal value contributed by a user in the round in which the user participates. With this scoring rule, we can also introduce the efficient all-play-first outcome as a subgame perfect Nash equilibrium for a large class

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<sup>2</sup>Harper et al. [4] show that factual questions have a higher archival value than discussion questions.

of complements information, namely any additive complements valuations. The proportional-share rule also retains from the best-answer rule the efficient outcome as the unique equilibrium in the case of substitutes information, for all substitutes valuation functions.

The reason the approval and proportional-share rules have better properties than best-answer for complements information is that the approval and proportional-share rules do not have the extreme winner-take-all property of the best answer rule.

Both new rules are able to achieve the all-play-first outcome as a subgame perfect Nash equilibrium for certain classes of complements valuations. The approval voting scoring rule introduces less efficient equilibrium for the case of substitutes information, however, it is a simple generalization of the best-answer scoring rule and seems easier to implement. Neither rule is quite optimal, in that neither rule isolates the efficient outcome as a unique equilibrium for all possible complements and substitutes valuation functions. We introduce three axioms, *anonymity*, *monotonicity* and *time-invariance* with which we show that there is no perfect rule in this sense, which provides support for the rules considered here.

## 1.1 Related Work

While we believe this to be the first work studying online question and answer forums in a game-theoretic light, there have been a number of related empirical studies.

Nam et al. [10] show that points are a factor in motivating users to participate in points-based question and answer forums. They study the Naver Knowledge-iN (KiN) system, the largest question and answer community in South Korea. They give a survey to 26 users of KiN and find that points are source of motivation for users, along with altruism, promoting personal businesses, learning and maintaining personal knowledge.

Adamic et al. [1] study contribution patterns in Yahoo! Answers and try to determine the extent to which certain statistical features such as response length effect the probability of an answer being chosen as a “best answer”. Yang et al. [14] conduct an empirical study of user behavior on *Taskcn*, a popular web-based knowledge sharing market in China (also known as a *Witkey*). These authors study user behavior over time, and find that a very small core of successful users manage to increase their win percentage over time and these users account for 20% of the winning answers. Strategic behavior is demonstrated, in that users learn to select tasks where they are competing against few opponents, and to increase their chances of winning. Users also, over time, select tasks with higher expected rewards.

In terms of game-theoretic analysis of other systems of human computation [13], prior work has presented a game-theoretic analysis of the *ESP game* [6] and the *PhotoSlap game* [5]. These are Games with a Purpose, games that are fun to play, with the added benefit that users are doing useful work in the process. While the game-theoretic analysis provided is specific to these games, these systems are similar to question and answer forums in that users are motivated by an artificial point system.

Closely related to the design of question and answer forums is the area of contest design and all-pay auctions [8, 9, 3]. Contests are situations in which multiple agents exert effort in order to win a prize. All agents bear the “cost” of the effort exerted regardless of whether they win a prize. Most of the literature in economics on contest design has focused on the case where agents compete in a single grand contest for a unique prize with complete information [11, 12, 2], though recent work has focused on more complex situations. For instance, Moldovanu and Sela [8] seek to understand how many prizes should be awarded and of what value in a single contest, where the principal has fixed resources, as to maximize his objective. Still, the agents do not build from each other’s solutions as in our model. The principal’s objective is typically to maximize the total effort exerted across all agents. Still, the agents do not build from each other’s solutions as in our model. Similarly, one can consider the benefit to the principal of designing a multi-round contest versus a single-round contest, but where each round remains simultaneous. For example, consider the scenario where the principal initially subdivides the users into a set of parallel sub-contests, with the winners of these sub-contests compete in the second round [9]. Still others consider how *agents choose* among a set of contests and subsequently exert effort to compete for prizes within the chosen contest [7].

## 2 Our Model

We focus on modeling how users participate in answering a single question posted by the asker. Each user  $j \in \{1, \dots, n\}$  has a unique piece of information  $I_j$ , with  $\mathcal{I} = \{I_1, I_2, \dots, I_n\}$ . Even though information is private, the fact that everyone possesses a piece of information out of  $n$  total pieces is common knowledge. The information aggregation process proceeds over a set of discrete rounds, with at most  $T > 1$  rounds, and closes in an earlier round if the value of the aggregate information reported up to and including that round exceeds a value threshold of the asker. Each user  $j$  observes the participation of other users and selects a single round in which to participate and aggregate  $I_j$  with the information reported so far.

Each piece of information is distinct but equivalent in terms of the value it provides to the asker, so that the asker's valuation function  $v : \{0, 1, \dots, n\} \rightarrow \mathbb{R}_{>0}$  depends only on the number of distinct pieces of information reported. We assume that  $v(0) = 0$  and  $v(j+1) > v(j)$  for all  $0 \leq j < n$ . In posting an answer as the only user active in a round, the value of the answer is  $v(\ell+1)$  where  $\ell$  users had previously posted an answer. By this, each user aggregates all previous answers in the processing of revealing his information. In posting an answer as one of a set of  $m$  users in a round, with  $\ell$  previous posts, the value of the answer of each of these users is  $v(\ell+m)$  to model the ability of the asker to aggregate all the information in this case of simultaneous revelation.

Depending on the nature of the question, the pieces of information related to the question may be complements or substitutes. For example, suppose the asker posts the question: "What should I do for a one-day visit to Boston?" The two pieces of information, "walk along the Freedom Trail" and "have lunch at Quincy Market (which is on the Freedom Trail)" are complements, because the value of knowing both pieces of information for the asker is higher than the sum of the values of only knowing a single piece of information. However, if the asker posts the question: "Where should I have lunch in Times Square?", the answers "Becco" and "Kodama" are substitutes for the asker, since the asker must choose between the two.

Let  $\delta_j = v(j) - v(j-1)$  for  $1 \leq j \leq n$ .

**Definition 2.1.** In the *complements* case, the asker's valuation function must satisfy  $\delta_j < \delta_{j+1}$  for all  $1 \leq j < n$ .

**Definition 2.2.** In the *substitutes* case, the asker's valuation function must satisfy  $\delta_j > \delta_{j+1}$  for all  $1 \leq j < n$ .

The asker has a private *value threshold*  $\theta$ , sampled uniformly on  $[0, v(n)]$ . The distribution from which  $\theta$  is sampled is common knowledge. The asker prefers a satisficing answer (with value at least  $\theta$ ) as soon as possible, and closes the question in the first round in which the value of an answer meets or exceeds the threshold. Upon closing the question the asker assigns points (any non-negative value, in general) to some subset of the users who have responded, according to a scoring rule. Based on this, each user seeks to maximize her expected score. Because we choose not to associate a cost with the participation of a user, it is without loss to consider only strategies in which a user submits an answer in some round.

Let  $b(t) \in \{0, \dots, n\}$  denote the number of pieces of information revealed up to and including round  $t \in \{0, 1, \dots, T\}$ .

**Lemma 2.3.** *The probability of stopping in round  $k$ , for  $k \geq 1$ , is  $P(k) = \frac{(v(b(k)) - v(b(k-1)))}{v(n)}$ .*

*Proof.* The probability of stopping in round  $k$  is the probability that  $\theta \leq v(b(k))$  and  $\theta > v(b(k-1))$ , which is just  $\frac{(v(b(k)) - v(b(k-1)))}{v(n)}$  for the uniform distribution.  $\square$

### 2.1 The Scoring Rules

In this paper, we examine the equilibrium behavior of this question and answer game under three different scoring rules.

**Best Answer Scoring Rule** The best answer rule models the method of assigning points currently used by Yahoo! Answers. In Yahoo! Answers, upon closing the question, the asker can select one answer as the

best answer and the associated user is then awarded some fixed number of points. Without loss of generality, we normalize the number of points awarded to 1.

When the asker closes the question because the value has reached the threshold, the asker awards a single point to the user  $i \in A$  that maximizes  $b(t_i)$ , where  $A$  is the set of agents who participated before the question closed. In other words, the best answer rule awards a single point to the user who provides the answer with the largest total value. If there is a tie for the answer with the largest total value, ties are broken uniformly at random. Given our assumption that users incorporate information from previous rounds into their answers, the best answer rule awards a point to the user who played in the last round before the question closed.

**Approval Voting Scoring Rule** Under the approval voting scheme, the asker can provide the same reward to each of  $k > 1$  users, where  $k < n$ . The number of winners,  $k$  is a design parameter. Note that if  $k = 1$ , this reduces to the best answer rule.<sup>3</sup> Under the approval voting scoring rule, the  $k$  users who provide the answers with the largest total values are rewarded. Given our assumption that users incorporate the information from previous rounds into their answers, the approval voting rule awards a point to the  $k$  most recent users.

In our model, we assume that the asker will always assign  $k < n$  winners if possible and the  $k$  most recent users to answer before the question is closed will each receive a point, with ties broken uniformly at random. In the special case in which the question is closed and less than  $k$  users have responded, these users each receive one point. When more than  $k$  users respond in the most recent round, then a subset of  $k$  winners is selected uniformly at random. Similarly, when less than  $k$  users (say  $k_1$ ) respond in the most recent round but more than  $k - k_1$  users responded in the previous round, then a subset of  $k - k_1$  users from the previous round are selected as winners uniformly at random.

**Proportional Share Scoring Rule** In the proportional-share scoring rule, the asker is given a fixed number of points that she can distribute. Without loss of generality, we normalize the total number of points to distribute to 1. We assume that the asker distributes the point according to her valuation function. More specifically, suppose the question closes after  $\ell \leq T$  active rounds and at each active round  $t \leq \ell$  there are  $n_t$  participants. In the proportional-share scoring rule, the asker distributes  $\frac{v(b(1))}{v(b(\ell))}$  equally among the  $n_1$  users participated in round 1, and, similarly, distributes  $\frac{v(b(t)) - v(b(t-1))}{v(b(\ell))}$  to the  $n_t$  users that participated in active round  $t > 1$ , where  $v(b(t))$  denotes the value of the items received at the end of round  $t$ .

Given that our game is one of multiple time periods and that users have information about the game play in previous time periods, we use the subgame perfect Nash equilibrium concept to analyze this game. We use the notion of an *active round* in our analysis. A round is considered active if at least one user participates in that round, in equilibrium, when  $\theta = 1$ .

## 3 Equilibrium Analysis

### 3.1 Best-Answer Rule

Let  $h^t$  denote the number of agents that have submitted an answer up to but not including round  $t$ . Consider the following strategy for agent  $i$ :

$$s_i(h^t) = \begin{cases} \text{play} & , \text{ if } t = T \\ \text{wait} & , \text{ if } t < T \end{cases}$$

We show that all players adopting this strategy is a unique subgame perfect Nash equilibrium (SPNE) for the best-answer rule and with complements information.

**Theorem 3.1.** *For any valuation function satisfying the complements condition, the unique subgame perfect Nash equilibrium under the best-answer rule is the strategy profile in which all players always play in the last round.*

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<sup>3</sup>It should be noted that Naver Knowledge-iN allows askers to select more than one best answer.

*Proof.* Consider an arbitrary subgame characterized by history  $h^t$  in which user  $i$  has not yet played. Assume at the start of this arbitrary subgame that  $m$  users have played thus far. Consider an arbitrary strategy  $s'$  of users  $\neq i$ . Let  $j$  denote the number of other users that play in the last round when  $i$  follows the prescribed strategy and plays in the last round. The expected payoff to agent  $i$  is  $P(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(m)}{v(n)}) \cdot \frac{1}{j+1}$ . Now consider a single deviation by user  $i$  in period  $t$ , where  $\ell$  denotes the number of other users that participate in round  $t$  under  $s'$ . We necessarily have  $\ell \leq n - j - 1$ . The expected payoff to agent  $i$  for *play* in this round is  $P(\frac{v(m)}{v(n)} < \theta < \frac{v(m+\ell+1)}{v(n)} | \theta > \frac{v(m)}{v(n)}) \cdot \frac{1}{\ell+1}$ . Since the valuation function satisfies the complements condition, we know that  $P(\theta > \frac{v(n-j-1)}{v(n)}) \cdot \frac{1}{j+1} > P(\frac{v(m)}{v(n)} < \theta < \frac{v(m+\ell+1)}{v(n)}) \cdot \frac{1}{\ell+1}$  for any positive integer values of  $j$  and  $\ell$ . Therefore we have  $P(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(m)}{v(n)}) \cdot \frac{1}{j+1} > P(\frac{v(m)}{v(n)} < \theta < \frac{v(m+\ell+1)}{v(n)} | \theta > \frac{v(m)}{v(n)}) \cdot \frac{1}{\ell+1}$  for any positive integer values of  $j$  and  $\ell$ .

This establishes that whatever the strategy of other players forward from any subgame, the strict best response of player  $i$  is to follow the prescribed strategy. This establishes that playing in the last round is a unique SPNE and completes the proof.  $\square$

Now consider the following strategy for player  $i$ :

$$s_i(h^t) = \begin{cases} \text{play} & \text{if } t = 1 \\ \text{no available action} & \text{if } t > 1 \end{cases}$$

**Theorem 3.2.** *For any valuation function satisfying the substitutes condition, the unique subgame perfect Nash equilibrium under the best answer rule is the strategy profile in which all users play in the first round.*

*Proof.* Consider the first round. Fix an arbitrary strategy  $s'$  of users  $\neq i$  and let  $j$  denote the number of other users that play in round  $t = 1$ . The expected payoff to agent  $i$  under the prescribed strategy is  $\frac{v(j+1)}{v(n)} \cdot \frac{1}{j+1}$ . Now consider a single deviation by user  $i$  in period 1, where now instead of playing in round 1, player  $i$  participates in a later round. Suppose that  $\ell$  other users participate in this round (with  $\ell \leq n - j - 1$  necessarily, under  $s'$ .) Furthermore assume that  $m$  users have participated before this round, under  $s'$  (where  $m \geq j$ ). The expected payoff to agent  $i$  is  $\frac{p'}{\ell+1}$  where  $p'$  is the probability the threshold is first reached in this later round, conditioned on  $s'$ . In particular,  $p' \leq \frac{v(\ell+1+m)-v(m)}{v(n)}$ . Because of the substitutes assumption,  $\frac{v(j+1)-v(0)}{v(n)} \cdot \frac{1}{j+1} > \frac{v(\ell+1+m)-v(m)}{v(n)} \cdot \frac{1}{\ell+1}$ , for any values of  $j$ ,  $\ell$ , and  $m$ , and the payoff to player  $i$  is always greater if she plays in the first round.

This establishes that it is a strict best response for agent  $i$  to play in the first round whatever the strategies of the other players, and establishes this as the unique SPNE.  $\square$

### 3.2 Approval Rule

Recall that the approval rule is parameterized by  $k \in \{2, \dots, n - 1\}$ , where  $k$  are the number of points awarded in total.

In considering the case of complementary valuations, we first establish a useful characterization result. The proof of this theorem is deferred to the Appendix and it is obtained via strong induction on the number of agents in the last two active rounds,

**Theorem 3.3.** *All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the approval voting rule (with  $k > 1$ ) for any complements valuations.*

Consider the following partially-specified strategy for agent  $i$ :

$$s_i^\ell(h^t) = \begin{cases} \text{play} & \text{if } t = \ell \text{ and no one else has played} \\ \text{wait} & \text{if } t < \ell \text{ and no one else has played} \end{cases}$$

We first show that all players adopting this strategy profile is a subgame perfect Nash equilibrium for any value of  $\ell \in \{1, \dots, T\}$  for complements valuations that are not too complementary.

**Theorem 3.4.** *For any valuation function that satisfies the complements condition and  $\frac{v(n)-v(n-1)}{v(n)} \leq \frac{k}{n}$ , all users playing  $s_i^\ell$ , for any value of  $\ell$ , is a subgame perfect Nash equilibrium under the approval voting scoring rule for  $k > 1$  winners.*

*Proof.* If all users  $i$  play strategy  $s_i^\ell$ , the on-the-path behavior is for all users to play in round  $\ell$  and the expected payoff to each player is  $\frac{k}{n}$ . Suppose that a player  $j$  deviates and goes earlier, we know from Theorem 3.3, that the remaining  $n - 1$  players will all play in the same round in any strategy profile that is a SPNE. Therefore, the expected payoff to a user  $j$  who deviates and goes earlier is less than  $\frac{1}{n}$  in the complements case. Thus this deviation is not profitable.

Now suppose that a user  $j$  deviates by going later. By deviating to a later round, the expected payoff is  $\frac{v(n)-v(n-1)}{v(n)}$ , which is at most  $\frac{k}{n}$  by assumption. This completes the proof, and establishes that all players following  $s_i^\ell$  is a SPNE for any  $\ell$ .  $\square$

**Theorem 3.5.** *For any valuation function that satisfies the complements condition and  $\frac{v(n)-v(n-1)}{v(n)} > \frac{k}{n}$ , the unique subgame perfect Nash equilibrium under the approval rule is for all users to play last, in the on-the-path play.*

*Proof.* Given Theorem 3.3, it suffices to consider strategy profiles in which all players participate in the same round in every subgame. First suppose that all players play in round  $\ell < T$  in equilibrium. The payoff to any player is  $\frac{k}{n}$ . Deviation to a later round obtains  $\frac{v(n)-v(n-1)}{v(n)} > \frac{k}{n}$ , and is profitable. Consider  $\ell = T$ . Suppose player  $j$  deviates to an earlier round. Theorem 3.3 tells us that all remaining players will participate in the same round. Therefore the expected payoff to  $j$  is  $\frac{v(1)}{v(n)} < \frac{1}{n} < \frac{k}{n}$ , where the first inequality is from the complements property. Thus, the unique subgame perfect Nash equilibrium is for all users to play last.  $\square$

Thus we know for the case of complements valuations, if the valuation function is “very complementary”, the equilibrium results are the same as with the best answer rule, but for less extreme valuations, we have all pooling equilibrium, or in other words, all users play in the same round, for any round, in equilibrium.

Now we shift attention to the case of substitutes valuations. Similar to the complements case, we get a complete characterization result for the case of substitutes valuations. This result is established via strong induction on the number of agents in the last two active rounds. The proof is deferred to the Appendix.

**Theorem 3.6.** *All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the approval voting rule (with  $k > 1$ ) for any substitutes valuations.*

Again consider the partially-specified strategy profile  $s_i^\ell(h^t)$  for some  $\ell \in \{1, \dots, T\}$ . The following observation is for valuations that are not too substitutable.

**Theorem 3.7.** *For any valuation function that satisfies the substitutes condition and  $\frac{v(1)}{v(n)} \leq \frac{k}{n}$ , all users playing  $s_i^\ell$ , for any value of  $\ell$ , is a subgame perfect Nash equilibrium under the approval rule for  $k > 1$  winners.*

*Proof.* If all players play strategy  $s_i^\ell$  the expected payoff to each player is  $\frac{k}{n}$ . Consider a player  $j$  who deviates and goes earlier. Theorem 3.6 tells us that the remaining  $n - 1$  players will play in the same round in any SPNE. Therefore, we know that the expected payoff of a user  $j$  is  $\frac{v(1)}{v(n)} \cdot 1 < \frac{k}{n}$  by assumption. Thus this deviation is not profitable. Considering the other players, the expected payoff of such a player, conditioned on making it to the next round, for the stipulated strategy, is  $\frac{k}{n-1}$ .

Now consider a deviation by user  $j$  to a later round. The expected payoff is  $\frac{v(n)-v(n-1)}{v(n)} < \frac{1}{n} < \frac{k}{n}$ , where the first inequality is by the substitutes property. This completes the proof, and establishes that all players following  $s_i^\ell$  is a SPNE, for any  $\ell$ .  $\square$

We now establish that the approval voting rule isolates the most efficient outcome for “very substitutable” valuations.

**Theorem 3.8.** *For any valuation function that satisfies the substitutes condition and  $\frac{v(1)}{v(n)} > \frac{k}{n}$ , the unique subgame perfect Nash equilibrium under the approval rule is for all users to play first, in the on-the-path play.*

*Proof.* Given Theorem 3.6, it suffices to consider strategy profiles in which all players participate in the same round in every subgame. First suppose all players play in round  $\ell > 1$ . The payoff to any player is  $\frac{k}{n}$ . The expected payoff to a player  $j$  who deviates to an earlier round is at least  $\frac{v(1)}{v(n)}$ , and so greater than  $\frac{k}{n}$  by assumption.

Consider  $\ell = 1$ . The expected payoff to a player who deviates to a later round is  $\frac{v(n)-v(n-1)}{v(n)} < \frac{1}{n} < \frac{k}{n}$ , where the first inequality follows by the substitutes property, and this is not profitable. This completes the proof.  $\square$

In other words, for the case of substitutes, we get that if the valuation function is very substitutable, the unique subgame perfect Nash equilibrium is for all players to go first. For other substitutes valuations, every pooling profile is a subgame perfect Nash equilibrium.

### 3.3 Proportional Share Scoring Rule

The approval voting scoring rule can introduce the most efficient equilibrium for certain complements valuations, but also the least efficient equilibrium for certain substitutes valuations.

To this end, we consider the proportional share scoring rule to see whether it is possible to introduce the most efficient equilibrium for the case of complements valuations without introducing the least efficient equilibrium for the case of substitutes valuations.

Similar to the approval voting scoring rule, we get a strong characterization result for the proportional share scoring rule. For this characterization result, we assume additional structure on complements valuations, namely additive complements. Our characterization result holds for all additive complements valuations, i.e. for all  $c > 0$ . Theorem 3.10 is established by strong induction on the number of players in the last two active rounds and the proof is deferred to the Appendix.

**Definition 3.9.** We say that a valuation function exhibits additive complements if and only if  $v(1) = c$ ,  $v(2) - v(1) = 2c$ ,  $v(3) - v(2) = 3c$ , ...,  $v(n) - v(n-1) = nc$  for any  $c > 0$ . In other words,  $v(1) = c$ ,  $v(2) = 3c$ , ...,  $v(n-1) = \frac{n(n-1)c}{2}$ ,  $v(n) = \frac{(n+1)nc}{2}$ .

**Theorem 3.10.** *All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the proportional share for any additive complements valuations.*

Thus for the case of additive complements, it suffices to consider cases where agents play in the same round in every subgame.

**Lemma 3.11.** *For any additive complements valuation, the proportional share rule, and any strategy profile in which all users playing in the same round: (a) if  $\frac{v(1)}{v(n)} \leq 1 - \sqrt{\frac{n-1}{n}}$ , a user cannot profitably deviate by going in an earlier round, and (b) if  $\frac{v(n-1)}{v(n)} \geq 1 - \sqrt{\frac{1}{n}}$ , a user cannot profitably deviate by going in a later round.*

*Proof.* Consider the strategy profile consisting of all users going in the same round. The expected payoff of each user is  $\frac{1}{n}$ . Consider a user who deviates by playing in a later round. The expected payoff of such a user is  $(1-p) \cdot (1-p)$ , where  $p = \frac{v(n-1)}{v(n)}$ . In order for this deviation not to be profitable, we need  $(1-p)^2 \leq \frac{1}{n}$ , or equivalently,  $p \geq 1 - \sqrt{\frac{1}{n}}$ . Now consider a user who deviates by playing in an earlier round. Theorem 3.10 tells us that the remaining  $n-1$  players will play in the same round in an SPNE. Therefore, the expected payoff of such a user is  $p + (1-p) \cdot p$ , where  $p = \frac{v(1)}{v(n)}$ . In order for this deviation not to be profitable, we need  $p + (1-p) \cdot p \leq \frac{1}{n}$ , or equivalently,  $p \leq 1 - \sqrt{\frac{n-1}{n}}$ .  $\square$

In fact, for  $n \geq 3$ , we have pooling equilibrium for all rounds. For the case of  $n = 2$ , the only equilibrium is in which both players play first.

**Theorem 3.12.** *For the case of additive complements, with  $n \geq 3$ , the set of SPNE involve all users playing in the same round for any round.*

*Proof.* From Theorem 3.10, it suffices to consider strategy profiles where all users play in the same round. From Lemma 3.11, it suffices to show that  $\frac{v(1)}{v(n)} \leq 1 - \sqrt{\frac{n-1}{n}}$  and  $\frac{v(n-1)}{v(n)} \geq 1 - \sqrt{\frac{1}{n}}$ . Since  $n(n+1)^2 < (n-1)(n+2)^2$  for all  $n \geq 3$ , we know that  $\frac{n-1}{n} < (\frac{(n-1)(n+2)}{n(n+1)})^2$  and  $\sqrt{\frac{n-1}{n}} < \frac{n(n+1)-2}{n(n+1)} = 1 - \frac{2}{n(n+1)}$ , so  $\frac{2}{n(n+1)} < 1 - \sqrt{\frac{n-1}{n}}$ , and equivalently  $\frac{v(1)}{v(n)} < 1 - \sqrt{\frac{n-1}{n}}$  for additive complements. Since  $(n+1)^2 > 4n$  for all  $n > 1$ , we know that  $\frac{1}{n} > (\frac{2}{n+1})^2$  and equivalently  $\frac{1}{n} > (1 - \frac{v(n-1)}{v(n)})^2$  for additive complements, and so  $\sqrt{\frac{1}{n}} > 1 - \frac{v(n-1)}{v(n)}$  and  $\frac{v(n-1)}{v(n)} > 1 - \sqrt{\frac{1}{n}}$ .  $\square$

In fact, one can also see from Lemma 3.11 that when  $\frac{v(1)}{v(n)} > 1 - \sqrt{\frac{n-1}{n}}$  and  $\frac{v(n-1)}{v(n)} < 1 - \sqrt{\frac{1}{n}}$  hold there are no pooling SPNE and thus there must be separating equilibrium for complements valuations that satisfy these conditions under the proportional share scoring rule. An example of valuation function that has a separating equilibrium under the proportional share scoring rule is as follows:  $v(1)/v(3) = 0.19$ ,  $v(2) - v(1)/v(3) = 0.21$ ,  $v(3) - v(2)/v(3) = 0.6$ .

Finally we show that the efficient “all-going-first” equilibrium is preserved (and isolated) for the substitutes case under the proportional-share scoring rule.

**Theorem 3.13.** *For any valuation that satisfies the substitutes condition, the unique subgame perfect Nash equilibrium under the proportional share scoring rule is the strategy profile consisting of all users going first in every subgame.*

*Proof.* Consider the strategy profile in which all players play first. Lemma 3.11 tells us that if all users are going in the first round, no user has incentive to deviate if and only if the valuation function satisfies the condition:  $\frac{v(n-1)}{v(n)} \geq 1 - \sqrt{\frac{1}{n}}$ , which is always satisfied by any valuation function that satisfies the substitutes condition. Now consider the situation where all users go in the same round, that is not the first round. The expected payoff of a user who deviates by playing in an earlier round is *at least*  $\frac{v(1)}{v(n)}$ , regardless of the game play of other agents. We know that  $\frac{v(1)}{v(n)} > \frac{1}{n}$ , and thus a player always wants to deviate and go earlier. Consider any strategy profile in which there are two or more active rounds. Suppose that  $j$  users play in the last active round. The expected payoff of a user who participates in the last active round is  $(1 - \frac{v(n-j)}{v(n)}) \cdot \frac{v(n) - v(n-j)}{jv(n)}$ . Consider the expected payoff of a user in the last round who deviates by going in the first active round. Suppose that  $i$  users participate in the first active round, including the user who deviated. Her expected payoff is *at least*  $\frac{v(i)}{v(n)} \cdot \frac{1}{i}$ , regardless of the game play that follows. For any valuation function that satisfies the substitutes condition, we know that  $\frac{v(i)}{i} \geq \frac{v(n) - v(n-j)}{j}$ , so  $\frac{v(i)}{iv(n)} > (1 - \frac{v(n-j)}{v(n)}) \cdot \frac{v(n) - v(n-j)}{jv(n)}$ . Thus no strategy profile in which there are two or more active rounds can be a Nash equilibrium.  $\square$

## 4 An Axiomatic Treatment

The best answer rule isolated the most efficient (all play first) equilibrium for the case of substitutes but isolated the least efficient equilibrium (all play last) for the case of complements. In comparison, both the approval and proportional rules are successful in isolating the *most* efficient equilibrium for certain complements valuations. By tuning parameter  $k$ , the approval rule can enable this for a larger class of complements valuations than the proportional share rule. Still, the least efficient equilibrium is retained for some complements valuations in both rules. Another consideration is that the approval rule, but not the proportional share rule, also introduces the least efficient equilibria for the case of substitutes.

These results beg the question: *Does there exist a scoring rule that isolates the best possible equilibrium for all asker valuations?* To answer this question (in the negative) we introduce three reasonable axioms for scoring rules in this context, and prove that no rule can meet these axioms and always isolate the efficient equilibrium, for all substitutes and complements valuations.

In building some intuition, consider the following three possible rules:

1. “Pay you only if you go first”: In this scoring rule, the asker pays each user who goes in the first round  $\frac{1}{j}$ , where  $j$  is the number of users who participate in the first round, and pays everyone else 0.

2. “Second-to-last”<sup>4</sup>: In this rule, the asker pays each user who goes in the second to last active round  $\frac{1}{j}$ , where  $j$  is the number of users who participate in the round and pays everyone else 0. If there is only one active round, players each receive  $\frac{1}{j}$ , where  $j$  is the number of users who participate in that active round.
3. “Uniform”: The asker pays each user  $\frac{1}{j}$  regardless of which round she participates in, where  $j$  is the number of users who get their information in before the question closes.

In the case of the first and the third rules, playing first is a dominant strategy, so the only subgame perfect Nash equilibria. In the first rule, an agent’s expected payoff of playing in the first round is strictly positive whereas it is zero for playing in a later round. In the third rule, we note that for a fixed value of  $\theta$ , an agent is paid the same regardless of which round he plays, as long as she gets her answer in. Therefore, an agent will always want to play first.

In the case of the second rule, the “all-going-first” strategy profile is a subgame perfect Nash equilibrium because any player that deviates and goes later will receive a payoff of 0, for any value of  $\theta$ . However, this is not a unique equilibrium.<sup>5</sup>

Let us now introduce three axioms that we consider natural for this problem. For this, let a *configuration*  $\vec{c} = (t_1, t_2, \dots, t_n)$ , denote a realization of play in which user  $i$  participates in round  $t_i$  (unless the question closes before that round.) For example, the configuration corresponding to an equilibrium strategy profile defines the rounds in which each agent plays for  $\theta = 1$ .

Let  $\vec{p}(\vec{c}, \theta) = (p_1, p_2, \dots, p_n)$  denote the expected payoff to each player for configuration  $\vec{c}$  and threshold  $\theta$ , as induced by a scoring rule (where the expectation is taken with respect to any randomization within the rule). Let  $b_{\text{first}}(\vec{c}, t)$  and  $b_{\text{last}}(\vec{c}, t)$  denote the total number of answers submitted in configuration  $\vec{c}$  after one of the agents (if any) plays in round  $t$  and at the end of round  $t$ , respectively. Adopt  $v(b(t))$  as shorthand for  $v(\vec{c}, b_{\text{last}}(t))$ , and so the total value that accrues by the end of round  $t$ .

1. **Anonymous**: We say that a scoring rule is anonymous if for all asker valuation functions and any configuration  $\vec{c}_1 = (t_1, t_2, \dots, t_n)$ , any threshold  $\theta$ , and any permutation  $\sigma$ , where  $\vec{c}_2 = \sigma(\vec{c}_1)$ , we have that  $\vec{p}(\vec{c}_2, \theta) = \sigma(\vec{p}(\vec{c}_1, \theta))$ .
2. **Time-Invariance**: We say that a scoring rule is time invariant if for all asker valuation functions and any pair of configurations  $\vec{c}_1 = (t_1, t_2, \dots, t_n)$ ,  $\vec{c}_2 = (s_1, s_2, \dots, s_n)$  such that  $s_i - t_i = d$  for all  $1 \leq i \leq n$ , for some integer  $d$ , then  $p_i(\vec{c}_1, \theta) = p_i(\vec{c}_2, \theta)$  for all  $1 \leq i \leq n$ .
3. **Value-Monotonic**: We say that a scoring rule is value monotonic if there exists an  $\alpha \in [0, 1]$  such that, for every asker valuation function, we have
  - (a) For every configuration, every pair of players  $i, j$ , and every  $\theta > \{v(b(t_i - 1)), v(b(t_j - 1))\}$ , then  $v'_\alpha(\vec{c}, t_i) \geq v'_\alpha(\vec{c}, t_j) \Rightarrow p_i(\vec{c}, \theta) \geq p_j(\vec{c}, \theta)$ , and
  - (b) For every configuration  $\vec{c}$ , every pair of players  $(i, j) \in \arg \max_{(i, j)} [v'_\alpha(\vec{c}, t_i) - v'_\alpha(\vec{c}, t_j)]$ , there exists a  $\theta > \{v(b(t_i - 1)), v(b(t_j - 1))\}$  for which  $v'_\alpha(\vec{c}, t_i) > v'_\alpha(\vec{c}, t_j) \Rightarrow p_i(\vec{c}, \theta) > p_j(\vec{c}, \theta)$ ,

where *contributed value*  $v'_\alpha(\vec{c}, t)$  is defined to be either  $v(b_{\text{last}}(\vec{c}, t)) - \alpha \cdot v(b_{\text{last}}(\vec{c}, t) - 1)$  or  $v(b_{\text{first}}(\vec{c}, t)) - \alpha \cdot v(b_{\text{first}}(\vec{c}, t) - 1)$ .

To give examples, if  $\alpha = 1$  and  $v'_\alpha(\vec{c}, t)$  is defined in terms of  $b_{\text{first}}$ , then value monotonicity insists on properties on the payoff to a player that depend on the marginal value contributed as though the player were to play first in the round in which it plays. On the other hand, if  $\alpha = 0$  and  $v'_\alpha(\vec{c}, t)$  is defined in terms of  $b_{\text{last}}$ , then value monotonicity insists on properties on the payoff to a player that depend on comparing the total value at the end of the round in which the player submits an answer. Essentially, the value-monotonicity

<sup>4</sup>(due to Yoav Wilf)

<sup>5</sup>All users playing in round  $\ell$ , for all  $1 < \ell < n$ , is a subgame perfect Nash equilibrium. A deviation of playing later yields payoff of 0, as opposed to  $\frac{1}{n}$ . If a user deviates and participates in round  $\ell' < \ell$ , and the remaining  $n - 1$  players respond with  $n - 2$  players participating in round  $\ell' + 1$ , followed by the last player in round  $\ell' + 2$ , this is not a profitable deviation. It should be noted that response of the  $n - 1$  players is a weak best response to the deviation.

axioms seeks to be agnostic about whether the appropriate “measure” of value contributed is marginal or cumulative ( $\alpha = 1$  or  $0$ ) and whether a player is considered to go first in a round or last in a round.

Condition 3 (a) is a weak monotonicity property while condition 3 (b) insists on strong monotonicity for at least a pair of players ( $i, j$ ) for which the difference on contribution is the greatest and for at least some threshold,  $\theta$ . Note that 3 (b) holds vacuously for a configuration  $\vec{c}$  in which all players play in the same round because contributed value  $v'_\alpha(\vec{c}, t_i) = v'_\alpha(\vec{c}, t_j)$  for all  $i, j$  and any definition of  $v'$  and  $\alpha$ .

A scoring rule is value monotonic if 3 (a) and (b) hold for some choice of  $\alpha$  and some choice of  $v'_\alpha$ , which can be alternatively constructed in terms of  $b_{\text{first}}$  or  $b_{\text{last}}$ . A scoring rule is *not* value monotonic if there is no such  $\alpha$  and  $v'_\alpha$  combination under which it meets conditions 3 (a) and 3 (b) for all valuation functions and all configurations.

**Remark 4.1.** *The “pay-you-if-you-go-first” scoring rule satisfies anonymity, but violates time-independence and value-monotonicity.*

*Proof.* The payoffs under this rule are indifferent to permutations of the configuration, however the payoffs are not indifferent to shifts in time. All users participating in round  $\ell = 1$  will lead to a payoff of  $\frac{1}{n}$  for each. All users participating in round  $\ell > 1$  will lead to a payoff of 0 for each.

Now we consider value-monotonicity. Consider  $\theta = 1$  and the configuration  $\vec{c}$  in which player  $i$  plays in the first round and all other players play in the second round. In this scenario, player  $i$  will receive a payoff of 1, while all other players receive a payoff of 0. However, for any complements valuation,  $v(1) < v(j) - v(j-1) \leq v(j) - \alpha v(j-1)$  for any  $\alpha \in [0, 1]$ , any  $j > 1$ . From this, we have  $v'_\alpha(\vec{c}, t_i) = v(1) - \alpha v(0) = v(1) < v(j) - \alpha v(j-1) \leq v'_\alpha(\vec{c}, t_r)$ , for any player  $r$  to play in the second round, where the first equality and the final inequality both hold irrespective of whether  $v'_\alpha$  is defined on  $b_{\text{last}}$  or  $b_{\text{first}}$ .  $\square$

**Remark 4.2.** *The “second-to-last” scoring rule satisfies time-independence and anonymity, but violates value-monotonicity.*

*Proof.* The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time. Now we consider value-monotonicity. Consider  $\theta = 1$  and the configuration  $c$  in which player  $i$  plays in the first round and all other players play in the second round. In this scenario, player  $i$  will receive a payoff of 1, while all other players receive a payoff of 0. The rest of the proof follows according to that of Remark 4.1.  $\square$

**Remark 4.3.** *The uniform scoring rule satisfies anonymity and time-independence, but violates value-monotonicity.*

*Proof.* The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time.

Now we consider value-monotonicity and specifically 3 (b). Consider a configuration  $\vec{c}$  in which each player plays in a separate round. Consider any complements valuation function and players ( $i, j$ ) maximizing the difference in contributed value. For any definition of adjusted value, we must have  $v'_\alpha(\vec{c}, t_i) > v'_\alpha(\vec{c}, t_j)$  since the players play in different rounds; e.g., if  $\alpha = 0$  and  $v'_\alpha$  is defined on  $b_{\text{last}}$  then this is the total value in a round. But uniform assigns the same score to every player who answers before the question closes, and so there is no  $\theta$  that allows both  $i$  and  $j$  to play for which  $i$ 's payoff is higher than  $j$ 's in configuration  $\vec{c}$ .  $\square$

We can revisit the best answer, approval and proportional-share scoring rules introduced earlier and consider them from this axiomatic perspective.

**Remark 4.4.** *The best answer scoring rule satisfies anonymity, time-independence, and value-monotonicity.*

*Proof.* The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time.

Now we consider value-monotonicity for  $\alpha = 0$  and  $v'_\alpha$  defined on  $b_{\text{last}}$  so that  $v'_\alpha(\vec{c}, t_i) = v(b(t_i))$ . Now, consider any configuration  $\vec{c}$ , any  $i, j$  and restrict attention w.l.o.g. to  $\theta$  for which both  $i$  and  $j$  play. For 3 (a), suppose  $v'_\alpha(\vec{c}, t_i) \geq v'_\alpha(\vec{c}, t_j)$ , so that we must have  $t_i \geq t_j$ . Then, we have  $p_i(\vec{c}, \theta) \geq p_j(\vec{c}, \theta)$  since if  $t_i > t_j$  then  $p_j(\vec{c}, \theta) = 0$ , while if  $t_i = t_j$  then the (expected) payoff is the same. Finally, if  $v'_\alpha(\vec{c}, t_i) > v'_\alpha(\vec{c}, t_j)$  and so  $t_i > t_j$ , we have  $p_i(\vec{c}, \theta) > p_j(\vec{c}, \theta)$  for  $\frac{v_i(b(t_i)-1)}{v(n)} < \theta \leq \frac{v_i(b(t_i))}{v(n)}$  because  $p_j(\vec{c}, \theta)$  is 0 and agent  $i$  competes to win with the other players (if any) in  $t_i$ .  $\square$

**Remark 4.5.** *The approval voting scoring rule satisfies anonymity, time-independence, and value-monotonicity.*

*Proof.* The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time.

Now we consider value-monotonicity for  $\alpha = 0$  and  $v'_\alpha$  defined on  $b_{\text{last}}$ , so that  $v'_\alpha(\vec{c}, t_i) = v(b(t_i))$ . Consider any configuration  $\vec{c}$ , any  $i, j$  and restrict attention w.l.o.g. to  $\theta$  for which both  $i$  and  $j$  play. For 3 (a), suppose  $v'_\alpha(\vec{c}, t_i) \geq v'_\alpha(\vec{c}, t_j)$ , so that we must have  $t_i \geq t_j$ . Then, if  $t_i > t_j$  then whenever  $j$  scores  $i$  also scores and thus  $p_j(\vec{c}, \theta) \leq p_i(\vec{c}, \theta)$ . If  $t_i = t_j$  then their (expected) payoff is the same. Then, consider a pair  $(i, j)$  that maximizes the difference  $v(b(t_i)) - v(b(t_j))$ , so that  $i$  plays in the first active round and  $j$  in the last active round. 3 (b) holds trivially for a configuration in which all players play in the same round, and so consider the case where  $t_i > t_j$ . Fix  $\theta = 1$ . For all valuation functions, we have if  $j$  scores then  $i$  scores. Moreover, conditioned on  $i$  scoring,  $j$ 's expected payoff is strictly less than 1 because  $j$  must compete for any remaining points  $\max(0, k - (n - n(t_i)))$  where  $n(t_i)$  play in round  $t_i$  with  $n(t_i)$  players, with  $n(t_i) > \max(0, k - (n - n(t_i)))$  since  $k < n$ . This completes the proof.  $\square$

**Remark 4.6.** *The proportional share scoring rule satisfies anonymity, time-independence and value-monotonicity.*

*Proof.* The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time.

Now we consider value-monotonicity, with  $\alpha = 1$  and  $v'_\alpha$  defined on  $b_{\text{first}}$ , so that  $v_\alpha(\vec{c}, t_i) = v(b(t_i - 1) - 1) - v(b(t_i - 1))$ . Consider any configuration  $\vec{c}$ , any  $i, j$  and restrict attention w.l.o.g., to any  $\theta$  for which both  $i$  and  $j$  play. First suppose that  $v(b(t_i - 1) - 1) - v(b(t_i - 1)) = v(b(t_j - 1) - 1) - v(b(t_j - 1))$ . For both substitutes and complements valuations, we must have  $t_i = t_j$ , from which the expected payoff is the same to  $i$  and  $j$  for any  $\theta$  that allows round  $t_i = t_j$ . This establishes part of what is required for 3 (a).

Second, suppose that  $v(b(t_i - 1) - 1) - v(b(t_i - 1)) > v(b(t_j - 1) - 1) - v(b(t_j - 1))$ . We establish 3 (a) and 3 (b) by showing that for *all*  $\theta$  in which both players play,  $i$ 's payoff is strictly greater than  $j$ 's. For complements valuations, we must have  $t_i > t_j$ . Conditioned on both agents playing, agent  $i$ 's expected payoff is  $\frac{v(b(t_i)) - v(b(t_i - 1))}{v(b(t_r))} \cdot \frac{1}{i}$ , where the game terminates in round  $r$  and agent  $j$ 's expected payoff is  $\frac{v(b(t_j)) - v(b(t_j - 1))}{v(b(t_r))} \cdot \frac{1}{j}$ . Then, we have  $\frac{v(b(t_i)) - v(b(t_i - 1))}{v(b(t_r))} \cdot \frac{1}{i} > \frac{v(b(t_j)) - v(b(t_j - 1))}{v(b(t_r))} \cdot \frac{1}{j}$  by complements. For substitutes valuations, we must have  $t_i < t_j$  and agent  $i$ 's expected payoff is  $\frac{v(b(t_i)) - v(b(t_i - 1))}{v(b(t_r))} \cdot \frac{1}{i}$ , compared to  $\frac{v(b(t_j)) - v(b(t_j - 1))}{v(b(t_r))} \cdot \frac{1}{j}$  for agent  $j$ . Then, we have  $\frac{v(b(t_i)) - v(b(t_i - 1))}{v(b(t_r))} \cdot \frac{1}{i} > \frac{v(b(t_j)) - v(b(t_j - 1))}{v(b(t_r))} \cdot \frac{1}{j}$  by substitutes.  $\square$

The following theorem establishes that we cannot have a scoring rule satisfy all three axioms and isolate the most efficient, “all-going-first” equilibrium as the unique subgame perfect Nash equilibrium.

**Theorem 4.7.** *There is no scoring rule that satisfies anonymity, time-independence and value monotonicity and isolates the all-going first equilibrium as the unique subgame perfect Nash equilibrium for all asker valuation functions.*

*Proof.* Assume otherwise, that is, assume that there exists a scoring rule that satisfies anonymity, time-independence and value monotonicity and isolates the all-going first equilibrium as the unique subgame perfect Nash equilibrium for all asker valuation functions. This means that if players all go in the first round, there is no profitable deviation (of going later). Now consider the strategy profile in which players all play in the  $t^{\text{th}}$  round where  $t > 1$ . First observe that there is no useful deviation by a player going later, because by time-independence, this profitable deviation would still be available for the  $t = 1$  strategy profile. Hence it suffices to consider deviations (say, by player 1) to an earlier round  $\ell < t$ . The payoff to player 1 from playing in round  $t$  is  $\frac{1}{n}$  by anonymity.

Consider a complements valuation, and let  $\vec{c}$  denote the configuration where 1 plays in round  $\ell$  and arbitrary play by the rest in subsequent rounds. By complements, we have  $v(1) - \alpha v(0) = v(1) < v(m) - v(m - 1) \leq v(m) - \alpha v(m - 1)$  for all  $\alpha \in [0, 1]$ , any  $m > 1$ . From this, then for *any* construction of contributed value  $v'_\alpha$ , i.e. any  $\alpha \in [0, 1]$  and with the use of  $b_{\text{first}}$  or  $b_{\text{last}}$ , we must have  $v'_\alpha(\vec{c}, \ell) < v'_\alpha(\vec{c}, t_j)$  for any  $j \neq 1$ . For the moment, consider  $\theta > \frac{v(n-1)}{v(n)}$ , i.e. high enough so that all players play. By value monotonicity, we must have  $p_1(\vec{c}, \theta) \leq p_j(\vec{c}, \theta)$  for all  $j \neq 1$ .

Denote by  $n$  an agent who plays in the last active round in  $\vec{c}$ . By complements, pair  $(1, n)$  maximizes the maximize the difference  $v'_\alpha(\vec{c}, t_i) - v'_\alpha(\vec{c}, t_j)$  for any construction of the contributed value function  $v'_\alpha$ . To see this: if  $\alpha = 0$  then  $v'_\alpha(\vec{c}, t_j)$  is minimized for  $j = 1$ , for either  $v'_\alpha(\vec{c}, t_j) = v(b_{\text{first}}(\vec{c}, t_j))$  or  $v'_\alpha(\vec{c}, t_j) = v(b_{\text{last}}(\vec{c}, t_j))$ . Similarly,  $v'_\alpha(\vec{c}, t_j)$  is maximized for an agent that plays in the last active round. Alternatively, if  $\alpha = 1$  then  $v'_\alpha(\vec{c}, t)$  is the marginal value contributed, either by the first answer or the last answer in round  $t$ , and by complements this is minimized for the first active round and maximized for the last active round. Clearly,  $(1, n)$  is also a maximizing pair for any  $\alpha \in (0, 1)$ . Then, by strict value monotonicity (3 b), there must exist a  $\theta > \frac{v(n-1)}{v(n)}$  such that  $p_1(\vec{c}, \theta) < p_n(\vec{c}, \theta)$ .

Given this, we know  $E_{\theta > \frac{v(n-1)}{v(n)}} [p_n(\vec{c}, \theta)] > E_{\theta > \frac{v(n-1)}{v(n)}} [p_1(\vec{c}, \theta)]$ , and that  $E_{\theta > \frac{v(n-1)}{v(n)}} [p_j(\vec{c}, \theta)] \geq E_{\theta > \frac{v(n-1)}{v(n)}} [p_1(\vec{c}, \theta)]$  for all  $j \neq 1$ . Since  $E_{\theta > \frac{v(n-1)}{v(n)}} [p_1(\vec{c}, \theta)] + E_{\theta > \frac{v(n-1)}{v(n)}} [p_2(\vec{c}, \theta)] + \dots + E_{\theta > \frac{v(n-1)}{v(n)}} [p_n(\vec{c}, \theta)] = 1$ , it must be that  $E_{\theta > \frac{v(n-1)}{v(n)}} [p_1(\vec{c}, \theta)] < \frac{1}{n}$ . The payoff to agent 1 for all  $\theta \leq \frac{v(n-1)}{v(n)}$  is at most 1, therefore its payoff for deviating earlier,  $E_\theta [p_1(\vec{c}, \theta)] \leq \Pr(\theta \leq \frac{v(n-1)}{v(n)}) + E_{\theta > \frac{v(n-1)}{v(n)}} [p_1(\vec{c}, \theta)]$ . In order for this deviation not to be profitable for player 1, we need  $E_\theta [p_1(\vec{c}, \theta)] < \frac{1}{n}$ . Setting  $\frac{v(n-1)}{v(n)} < \frac{1}{n} - E_{\theta > \frac{v(n-1)}{v(n)}} [p_1(\vec{c}, \theta)]$ , gives us the desired result. Therefore, there exists a sufficiently complementary valuation such that the payoff of any player for deviating earlier is less than playing in round  $t > 1$ . Hence all players going in round  $t$ , for any  $t > 1$ , is supported by a subgame perfect Nash equilibrium.  $\square$

## 5 Conclusions

We have introduced a simple, game-theoretic model of a question and answer forum such as Yahoo! Answers. We analyze the best-answer scoring rule, which models the scoring rule used in the Yahoo! Answers system, and show that it is effective with information items that are substitutes but enables only the least efficient outcome, in which every user plays in the very last round, in the case of complements. In considering the effect of different scoring rules on the equilibrium structure of the game, we have identified two scoring rules that lead to efficiency-improving changes in the equilibrium. Both the approval-voting rule and proportional-share rule can enable the most efficient equilibrium for certain complements valuations. The approval voting rule also introduces the least efficient equilibrium outcome for the case of substitutes information, for certain valuations, while the proportional share rule maintains the most efficient outcome as the unique equilibrium. The approval voting rule has a tunable parameter  $k$  that represents the tradeoff between the benefit of this scoring rule for the case of complements information and the disadvantage of introducing this scoring rule for the case of substitutes information.

Noticing that both the approval voting and proportional share rules have certain drawbacks, we took an axiomatic approach to describing scoring rules and found that there does not exist a rule that can isolate the “all-going-first” equilibrium and satisfy a set of “reasonable” axioms, such as anonymity, time independence and value monotonicity.

There are many avenues for future work. One could study variations on our simple model, such as answerers that have different valued pieces of information (from the asker side) and answerers that have overlapping information sets. Such extensions would remove symmetry of the answerers and move us toward a richer model. It would also be interesting to incorporate the fact that some users are partially motivated by altruistic reasons into our model. Another direction for future work would be to model the cost to the asker of combining information provided by multiple users in the same round, leading to the identification of scoring rules that promote “build” equilibrium where the user responses are optimally sequenced and build of each other. Another interesting extension is to consider information cascade effects, wherein one user’s response triggers another user to recall a new piece of information that would have not been available if not triggered by the first user. We believe that appropriate incentive design can help to improve the information quality in question and answer forums.

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# Appendix

## 6 Approval Voting Scoring Rule

**Theorem 3.3.** All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the approval voting rule (with  $k > 1$ ) for any complements valuations.

This result is established via strong induction on the total number of agents that play in the last two active rounds (in equilibrium).

**Lemma 6.1.** *No strategy profile (with at least two active rounds in equilibrium) that has  $l \leq k$  players in the last two active rounds can be a subgame perfect Nash equilibrium with the approval voting rule (with  $k > 1$ ) for any valuation function.*

*Proof.* Consider the subgame corresponding to the penultimate active round. Suppose that  $i$  agents play in this round and  $j$  play in the last active round. The expected payoff (conditioned on reaching this subgame) of an agent in the last active round is  $p_2 = \Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j+1} + \dots + \delta_n}{\delta_{n-j-i+1} + \dots + \delta_n}$ . By deviating to the penultimate active round, its expected payoff would be 1 since the agent is then sure to be one of the last  $k$  agents to play before the question closes. Therefore, the player can profitably deviate.  $\square$

From this, we can immediately establish the base case of our induction. The base case holds equally for both substitutes and complements valuations.

**Lemma 6.2.** *No strategy profile (with at least two active rounds in equilibrium) that has two players in the last two active rounds can be a subgame perfect Nash equilibrium under the approval voting scoring rule (with  $k > 1$ ) for any valuation function.*

*Proof.* Immediate from Lemma 6.1 since  $k > 1$ .  $\square$

Now we are ready for the inductive step. Define  $S_l$  as follows: *No strategy profile (with at least two active rounds in equilibrium) that has  $l$  players in the last two active rounds can be a subgame perfect Nash equilibrium with the approval voting rule (with  $k > 1$ ) for complements valuations.*

**Lemma 6.3.** *Given that statements  $S_2, \dots, S_l$  are true for  $l < n$ , then  $S_{l+1}$  is true.*

*Proof.* By Lemma 6.1 we can focus on the case of  $k < l$ . For ease of presentation, refer to the penultimate active round as round #1 and the last active round as round #2. Suppose  $i$  agents play in #1 and  $j$  in #2 (where  $i + j = l + 1$ ). For now, assume that  $i \geq 2$ . In all cases, consider the subgame (round #1) reached when all players are following the prescribed strategy, and condition on having reached this subgame.

Consider the case that  $j = 1$ . Since  $k < i + j$ , we can assume  $i \geq k$ . The expected payoff to an agent in #1 is  $p_1 \cdot \frac{k}{i} + p_2 \cdot \frac{k-1}{i}$  where  $p_1 = \Pr(\theta \leq \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j-i+1} + \dots + \delta_{n-j}}{\delta_{n-j-i+1} + \dots + \delta_n}$  and  $p_2 = \Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j+1} + \dots + \delta_n}{\delta_{n-j-i+1} + \dots + \delta_n}$ , and recognizing that the agent can compete for votes in the event that the question closes in #2. Consider a deviation to a later round by such an agent. By the inductive hypothesis, this would be a subgame with  $l$  or less players, and so all would play in the same round. The expected payoff would be  $p_3 = \Pr(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j} + \dots + \delta_n}{\delta_{n-j-i+1} + \dots + \delta_n}$ . For a SPNE we need  $p_1 \cdot \frac{k}{i} + p_2 \cdot \frac{k-1}{i} \geq p_3$  (eq. 6). The expected payoff to the agent in #2 is  $p_2$ . Deviating to #1 would bring this agent payoff  $\frac{k}{i+1}$  and so for a SPNE we need  $p_2 \geq \frac{k}{i+1}$  (eq. 5), which becomes  $\delta_n \geq (\delta_{n-i} + \dots + \delta_n) \cdot \frac{k}{i+1}$ . Eq. 6 becomes  $(\delta_{n-i} + \dots + \delta_{n-1}) \cdot \frac{k}{i} + \delta_n \cdot \frac{k-1}{i} \geq \delta_{n-1} + \delta_n$ , and so  $(\delta_{n-i} + \dots + \delta_{n-1}) \cdot k + \delta_n \cdot k \geq i \cdot \delta_{n-1} + (i+1) \cdot \delta_n$ , and so  $(\delta_{n-i} + \dots + \delta_n) \cdot k > (i+1) \cdot \delta_n$ , and a contradiction with (eq. 5).

Continuing, we can now assume  $j > 1$ , along with  $i \geq 2$ .

Consider the case  $i < k$  and  $j \geq k$ . The expected payoff to an agent in #1 is  $p_1$  and deviating to a later round brings  $p_3 \cdot \frac{k}{j+1}$  by a similar argument to above (and appeal to the inductive hypothesis.) For a SPNE, we need  $v(n-j) - v(n-j-i) \geq (v(n) - v(n-j-1)) \cdot \frac{k}{j+1}$ . But,  $\frac{v(n-j) - v(n-j-i)}{k} < \frac{v(n-j) - v(n-j-i)}{i} < \frac{v(n) - v(n-j-1)}{j+1}$  where the first inequality follows since  $i < k \leq j$  and the second by the complements property. Therefore a user in #1 has a profitable deviation.

Consider the case  $i < k$  and  $j < k$ , still with  $k < i + j$ . Now, the expected payoff to an agent in #1 is  $p_1 + p_2 \cdot \frac{k-j}{i}$ , and to preclude a useful deviation we need  $p_1 + p_2 \cdot \frac{k-j}{i} \geq p_3$  (eq. 1). By similar arguments to above, we also require  $p_2 \geq p_4 + p_5 \cdot \frac{k-j+1}{i+1}$  (eq. 2), where  $p_4 = \Pr(\theta \leq \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)})$  and  $p_5 = \Pr(\theta > \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)})$  to preclude a useful deviation by an agent in #2 to #1. Eq. 1 becomes  $(\delta_{n-j-i+1} + \dots + \delta_{n-j}) + (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k-j}{i} \geq \delta_{n-j} + \dots + \delta_n$ , or equivalently  $\delta_{n-j+1} + \dots + \delta_n \leq (\delta_{n-j-i+1} + \dots + \delta_{n-j-1}) + (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k-j}{i}$ . Eq. 2 becomes  $(\delta_{n-j-i+1} + \dots + \delta_{n-j+1}) + (\delta_{n-j+2} + \dots + \delta_n) \cdot \frac{k-j+1}{i+1} \leq \delta_{n-j+1} + \dots + \delta_n$ . Combining with Eq. 1 and canceling terms, this gives  $\delta_{n-j} + \delta_{n-j+1} + (\delta_{n-j+2} + \dots + \delta_n) \cdot \frac{k-j+1}{i+1} \leq (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k-j}{i}$ , but this is a contradiction because  $k < i + j$  and so  $\frac{k-j+1}{i+1} > \frac{k-j}{i}$ .

Consider the case  $i \geq k$  and  $j \geq k$ . The expected payoff to an agent in #1 is  $p_1 \cdot \frac{k}{i}$  and to be a SPNE we require  $p_1 \cdot \frac{k}{i} \geq p_3 \cdot \frac{k}{j+1}$ . For this, we need  $v(n-j) - v(n-j-i) \frac{k}{i} \geq (v(n) - v(n-j-1)) \frac{k}{j+1}$ , but  $\frac{v(n-j)-v(n-j-i)}{i} < \frac{v(n)-v(n-j-1)}{j+1}$  for any complements valuation.

Consider the case  $i \geq k$  and  $j < k$ . Considering a deviation by an agent in #1, we need  $p_1 \cdot \frac{k}{i} + p_2 \frac{(k-j)}{i} \geq p_3$ . This requires  $(\delta_{n-j-i+1} + \dots + \delta_{n-j}) \cdot \frac{k}{i} + (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k-j}{i} \geq \delta_{n-j} + \dots + \delta_n$ , and so  $(\delta_{n-j-i+1} + \dots + \delta_{n-j}) \cdot \frac{k}{i} + (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k-j}{i} > \delta_{n-j+1} + \dots + \delta_n$ , or equivalently,  $(\delta_{n-j-i+1} + \dots + \delta_{n-j}) \cdot \frac{k}{i} > (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{i+j-k}{i}$  (Eq. 3). Considering a deviation by an agent in #2, we also need  $p_2 \geq p_4 \cdot \frac{k}{i+1} + p_5 \cdot \frac{k-j+1}{i+1}$ . This requires  $\delta_{n-j+1} + \dots + \delta_n \geq (\delta_{n-j-i+1} + \dots + \delta_{n-j+1}) \cdot \frac{k}{i+1} + (\delta_{n-j+2} + \dots + \delta_n) \cdot \frac{k-j+1}{i+1}$ , and so  $\delta_{n-j+1} + \dots + \delta_n > (\delta_{n-j-i+1} + \dots + \delta_{n-j}) \cdot \frac{k}{i+1} + (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k-j+1}{i+1}$  (since  $j > 1$  and so  $\frac{k}{i+1} < \frac{k-j+1}{i+1}$ ), or equivalently,  $(\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{i+j-k}{i+1} > (\delta_{n-j-i+1} + \dots + \delta_{n-j}) \cdot \frac{k}{i+1}$ . We have a contradiction with Eq. 3, and so this cannot be part of a SPNE.

Finally, we must consider the case  $i = 1$ . From Lemma 6.4, we know this case cannot be a SPNE.  $\square$

**Lemma 6.4.** *No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and  $l$  players in the last active round can be a SPNE with the approval voting rule (with  $k > 1$ ) for any complements valuation function, given that no strategy profile is a SPNE when there are either (a) at most  $l$  players in the last two active rounds, or (b) exactly  $l + 1$  players in the last two active rounds with at least two players in the penultimate active round.*

*Proof.* Let  $i$  be the player who participates in the penultimate active round. We will establish this via strong induction on the number of rounds before  $T$  where agent  $i$  plays. The expected payoff is conditioned throughout on reaching the penultimate active round.

*Base Case:* No strategy profile with at least two active rounds in equilibrium in which there is one player in the penultimate active round and this round is in period  $T - 1$  can be a SPNE. Let  $j$  denote the number of players in the last active round. By Lemma 6.1 we can assume  $j \geq k$ . The expected payoff to player  $i$  is  $\Pr(\theta \leq \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-1)}{v(n)}) = \frac{\delta_{n-j}}{\delta_{n-j} + \dots + \delta_n}$ , and deviating to play in round  $T$  brings expected payoff (conditioned on reaching the penultimate active round) of  $\frac{k}{j+1}$ . In order for this to be part of a subgame perfect Nash equilibrium, we need  $\frac{\delta_{n-j}}{\delta_{n-j} + \dots + \delta_n} \geq \frac{k}{j+1}$ . But since  $(\delta_{n-j} + \dots + \delta_n) \cdot \frac{1}{j+1} \geq \delta_{n-j}$ , for the case of complements, we know that  $(\delta_{n-j} + \dots + \delta_n) \cdot \frac{k}{j+1} > \delta_{n-j}$  (for  $k > 1$ ), and thus this strategy profile cannot be a subgame perfect Nash equilibrium.

*Inductive Hypothesis:* No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round, and this round is  $r + 1$  periods before  $T$  can be a SPNE. Again, focus on the case that  $j \geq k$  (because otherwise we can appeal to Lemma 6.1.) Consider what happens when the player in the penultimate active round deviates and goes later. By assumption (a) in the statement of the lemma we know that there can be at most two active rounds in the resulting subgame, because otherwise the last two active rounds would include  $l$  or less players. Then, by the inductive case for  $r$  periods to go and by assumption (b) in the statement of the lemma, the only SPNE in that subgame involves all players playing in the same round. Then, by the same analysis as for the base case, a player in the penultimate active round can profitably deviate to play in the same round with the other  $j$  players. This completes the proof.  $\square$

**Theorem 3.6.** All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the approval voting rule (with  $k > 1$ ) for any substitutes valuations.

This result is established via strong induction on the total number of agents that play in the last two active rounds (in equilibrium). Recall that an active round is a round in which at least one agent plays. The base case is already established above.

Now we are ready for the inductive step. Define  $S_l$  as follows: *No strategy profile (with at least two active rounds in equilibrium) that has  $l$  players in the last two active rounds can be a subgame perfect Nash equilibrium with the approval voting rule (with  $k > 1$ ) for substitutes valuations.*

**Lemma 6.5.** *Given that statements  $S_2, \dots, S_l$  are true, for  $l < n$ , then  $S_{l+1}$  is true.*

*Proof.* By Lemma 6.1 we can focus on the case  $k < l$ . For ease of presentation, refer to the penultimate active round as round #1 and the last active round as round #2. Suppose  $i$  agents play in #1 and  $j$  in #2 (where  $i + j = l + 1$ ). For now, assume that  $i \geq 2$ . In all cases, consider the subgame (round #1) reached when all players are following the prescribed strategy, and condition on having reached this subgame.

Consider the case that  $j = 1$ . Since  $k < i + j$ , we can assume  $i \geq k$ . The expected payoff to the agent in #2 is  $p_2 = \Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j+1} + \dots + \delta_n}{\delta_{n-j-i+1} + \dots + \delta_n}$ . Deviating to #1 would bring this agent payoff  $\frac{k}{i+1}$ . But, we have  $p_2 = \frac{\delta_{n-j+1} + \dots + \delta_n}{\delta_{n-j-i+1} + \dots + \delta_n} < \frac{1}{i+1} < \frac{k}{i+1}$  and so a useful deviation, where the first inequality is by the substitutes property.

Now assume  $j > 1$ , along with  $i \geq 2$ .

Consider the case  $i < k$  and  $j > k$ . The expected payoff to an agent in #1 is  $p_1 = \Pr(\theta \leq \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j-i+1} + \dots + \delta_{n-j}}{\delta_{n-j-i+1} + \dots + \delta_n}$ . Consider a deviation to a later round by such an agent. By the inductive hypothesis, this would be a subgame with  $l$  or less players, and so all would play in the same round. The expected payoff would be  $\frac{k}{j+1} p_3$  with  $p_3 = \Pr(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j} + \dots + \delta_n}{\delta_{n-j-i+1} + \dots + \delta_n}$ . For a SPNE we need  $p_1 \geq \frac{k}{j+1} p_3$  (eq. 6). The expected payoff to an agent in #2 is  $\frac{k}{j} p_2$ . Consider a deviation to round #1 by such an agent. By the inductive hypothesis, the remaining  $j - 1 \geq k$  agents would all play in the same round and so the expected payoff for a deviation would be  $p_4 = \Pr(\theta \leq \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j-i+1} + \dots + \delta_{n-j+1}}{\delta_{n-j-i+1} + \dots + \delta_n}$  (since  $i + 1 \leq k$ ). For a SPNE we need  $\frac{k}{j} p_2 \geq p_4$  (eq. 5). Eq. 5 becomes  $\frac{\delta_{n-j+1} + \dots + \delta_n}{j} \geq \frac{\delta_{n-j-i+1} + \dots + \delta_{n-j+1}}{k}$ , and so  $\frac{\delta_{n-j+1} + \dots + \delta_n}{j} \geq \frac{\delta_{n-j-i+1} + \dots + \delta_{n-j}}{k}$ . Eq. 6 becomes  $\frac{\delta_{n-j-i+1} + \dots + \delta_{n-j}}{k} \geq \frac{\delta_{n-j} + \dots + \delta_n}{j+1}$ . For any substitutes valuation,  $\frac{\delta_{n-j+1} + \dots + \delta_n}{j+1} > \frac{\delta_{n-j+1} + \dots + \delta_n}{j}$ , so both eq. 5 and eq. 6 cannot hold simultaneously.

Consider the case  $i < k$  and  $j = k$ . To preclude a deviation by an agent in #1 we again need  $p_1 \geq \frac{k}{j+1} p_3$  (eq. 4). But now when considering a deviation by an agent in #2 to #1, we must also observe that it can also benefit from a vote coming from the question closing when the remaining  $k - 1$  agents answer. Moreover, when playing in #2 the agent doesn't need to compete for a vote. For SPNE, we need  $p_2 \geq p_4 + p_5 \cdot \frac{1}{i+1}$  (eq. 3), where  $p_5 = \Pr(\theta > \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j+2} + \dots + \delta_n}{\delta_{n-j-i+1} + \dots + \delta_n}$ . Eq. 3 becomes  $(\delta_{n-j+1} + \dots + \delta_n) \geq (\delta_{n-j-i+1} + \dots + \delta_{n-j+1}) + (\delta_{n-j+2} + \dots + \delta_n) \cdot \frac{1}{i+1}$ , and so  $(\delta_{n-j+1} + \dots + \delta_n) \geq (\delta_{n-j-i+1} + \dots + \delta_{n-j}) + (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{1}{i+1}$ , or in other words,  $(\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{i}{i+1} \geq \delta_{n-j-i+1} + \dots + \delta_{n-j}$ . Eq. 4 becomes  $\delta_{n-j-i+1} + \dots + \delta_{n-j} \geq (\delta_{n-j} + \dots + \delta_n) \cdot \frac{k}{k+1}$ , and so  $\delta_{n-j-i+1} + \dots + \delta_{n-j} \geq (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k}{k+1}$ . Observing that  $i < k$  and so  $\frac{i}{i+1} < \frac{k}{k+1}$  we see that eq. 3 and eq. 4 cannot hold simultaneously.

Consider the case  $i < k$  and  $j < k$ , still with  $k < i + j$ . Now, the expected payoff to an agent in #1 is  $p_1 + p_2 \cdot \frac{k-j}{i}$  (recognizing that the agent can compete for votes in the event that the question closes in #2), and to preclude a useful deviation to a later round by an agent in #1, we require  $p_1 + p_2 \cdot \frac{k-j}{i} \geq p_3$  (eq. 1). By similar arguments to above, we also require  $p_2 \geq p_4 + p_5 \cdot \frac{k-j+1}{i+1}$  (eq. 2) to preclude a useful deviation by an agent in #2 to #1. Eq. 1 becomes  $(\delta_{n-j-i+1} + \dots + \delta_{n-j}) + (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k-j}{i} \geq \delta_{n-j} + \dots + \delta_n$ , and so  $\delta_{n-j-i+1} + \dots + \delta_{n-j} > (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{i+j-k}{i}$ . Eq. 2 becomes  $\delta_{n-j+1} + \dots + \delta_n \geq (\delta_{n-j-i+1} + \dots + \delta_{n-j+1}) + (\delta_{n-j+2} + \dots + \delta_n) \cdot \frac{k-j+1}{i+1}$ , and so  $\delta_{n-j+1} + \dots + \delta_n > (\delta_{n-j-i+1} + \dots + \delta_{n-j}) + (\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k-j+1}{i+1}$  (since  $k < i + j$ ), or in other words,  $(\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{i+j-k}{i+1} > \delta_{n-j-i+1} + \dots + \delta_{n-j}$ . We have a contradiction, and eq. 1 and eq. 2 cannot hold simultaneously.

Consider the case  $i \geq k$  and  $j > 1$ . The expected payoff to an agent in #2 is upper-bounded by  $p_2 \cdot \frac{k}{j}$ , since it is  $p_2$  when  $j \leq k$  and  $p_2 \cdot \frac{k}{j}$  otherwise. By similar arguments to above, deviating to #1 will bring an

agent at least  $p_4 \cdot \frac{k}{i+1}$ , since in the case where  $j \leq k$  the expected payoff is  $p_4 \cdot \frac{k}{i+1} + p_5 \cdot \frac{k-j+1}{i+1}$ . So, in order to preclude a useful deviation we at least need  $p_2 \cdot \frac{k}{j} \geq p_4 \cdot \frac{k}{i+1}$ , or in other words,  $(\delta_{n-j+1} + \dots + \delta_n) \cdot \frac{k}{j} \geq (\delta_{n-j-i+1} + \dots + \delta_{n-j+1}) \cdot \frac{k}{i+1}$ . But  $\frac{\delta_{n-j+1} + \dots + \delta_n}{j} < \frac{\delta_{n-j-i+1} + \dots + \delta_{n-j+1}}{i+1}$  for substitutes valuations, and so we have a contradiction.

Finally, when  $i = 1$ , Lemma 6.6 tells us that this cannot be a SPNE.  $\square$

**Lemma 6.6.** *No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and  $l$  players in the last active round can be a SPNE with the approval voting rule (with  $k > 1$ ) for any substitutes valuation function, given that no strategy profile is a SPNE when there are either (a) at most  $l$  players in the last two active rounds, or (b) exactly  $l+1$  players in the last two active rounds with at least two players in the penultimate active round.*

*Proof.* Let  $i$  be the player who participates in the penultimate active round. We will establish this via strong induction on the number of rounds before  $T$  where agent  $i$  plays. The expected payoff is conditioned throughout on reaching the penultimate active round. Terms  $p_1, p_2, p_4$  and  $p_5$  are as defined in the proof of Theorem 3.6.

*Base Case:* No strategy profile with at least two active rounds in equilibrium in which there is one player in the penultimate active round and this round is in period  $T - 1$  can be a SPNE. Let  $j$  denote the number of players in the last active round. By Lemma 6.1 we can assume  $j \geq k$ . For  $j > k$  we have expected payoff of  $p_1$  and  $p_2 \cdot \frac{k}{j}$  to an agent in #1 and #2 respectively. To preclude a deviation by an agent in #2 to #1 we need  $p_2 \cdot \frac{k}{j} \geq p_4$ . For an agent deviating from #1 to round  $T$ , we need  $p_1 \geq \frac{k}{j+1}$ . The first equation becomes:  $(j+1-k) \cdot \delta_{n-j} \geq (\delta_{n-j+1} + \dots + \delta_n) \cdot k$  and the second becomes  $(\delta_{n-j+1} + \dots + \delta_n) \cdot k \geq j \cdot (\delta_{n-j} + \delta_{n-j+1})$ , which cannot hold simultaneously.

For  $j = k$ , we have expected payoff of  $p_1$  and  $p_2$  to an agent in #1 and #2 respectively. To prevent a deviation by an agent in #2 to #1 we need  $p_2 \geq p_4 + p_5 \cdot \frac{1}{i+1}$ . For an agent deviating from #1 to round  $T$ , her expected payoff becomes  $\frac{k}{k+1}$ , and to preclude a deviation we need  $p_1 \geq \frac{k}{k+1}$  (eq. 7). Then, Eq. 8 is the same as  $\delta_{n-j} \leq (\delta_{n-j+2} + \dots + \delta_n) \cdot \frac{k-2}{k-1}$ . Eq. 7 becomes  $\delta_{n-j} \geq (\delta_{n-j} + \dots + \delta_n) \cdot \frac{k}{k+1}$ , and it is impossible for both eq. 7 and eq. 8 to hold simultaneously.

*Inductive Hypothesis:* No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and this round is  $r+1$  periods before  $T$  can be a SPNE. Again, assume w.l.o.g. that  $j \geq k$ . Consider what happens when the player in the penultimate active round deviates and goes later. By assumption (a) in the statement of the lemma we know that there can be at most two active rounds in the resulting subgame, because otherwise the last two active rounds would include  $l$  or less players. Then, by the inductive case for  $r$  periods to go, and by assumption (b) in the statement of the lemma, the only SPNE in the subgame following the deviation to a later round involves all players playing in the same round. Then, by the same analysis as for the base case, either the player in the penultimate active round can profitably deviate later or a player in the last active round can profitably deviate to the penultimate active round.  $\square$

## 7 Proportional Share Scoring Rule

**Theorem 3.10** All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the proportional share for any additive complements valuations.

This result is established via strong induction on the total number of agents that play in the last two active rounds (in equilibrium).

**Lemma 7.1.** *No strategy profile (with at least two active rounds in equilibrium) that has exactly two players in the last two active rounds can be a subgame perfect Nash equilibrium under the proportional share scoring rule for any additive complements valuation function.*

*Proof.* Consider the subgame corresponding to the penultimate active round. The expected payoff (conditioned on reaching this subgame) of an agent in the last active round is  $\Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)})$ .

$\frac{v(n)-v(n-1)}{v(n)} = \frac{v(n)-v(n-1)}{v(n)-v(n-2)} \frac{v(n)-v(n-1)}{v(n)}$ . By deviating to the penultimate active round, this player's expected payoff becomes  $\frac{v(n)-v(n-2)}{v(n)-v(n-2)} \frac{v(n)-v(n-2)}{2v(n)}$ . In order for this to be an equilibrium, we need:  $\frac{(v(n)-v(n-1))^2}{v(n)(v(n)-v(n-2))} \geq \frac{(v(n)-v(n-2))^2}{2v(n)(v(n)-v(n-2))}$ , which is equivalent for additive complements to:  $4(2n)^2 \geq (2n + 2(n-1))^2$  or  $0 \geq 2n^2 - 4n + 1$ . The right hand side is greater than 0 for all  $n > 2$  since the roots of  $2n^2 - 4n + 1$  are  $\approx 0.3, 1.7$ . Therefore, the player in the last active round can profitably deviate.  $\square$

Now we are ready for the inductive step. Define  $S_l$  as follows: *No strategy profile (with at least two active rounds in equilibrium) that has exactly  $l$  players in the last two active rounds can be a subgame perfect Nash equilibrium under the proportional share scoring rule for any additive complements valuations.*

**Lemma 7.2.** *Given that statements  $S_2, \dots, S_l$  are true for  $l < n$ , then  $S_{l+1}$  is true.*

*Proof.* For ease of presentation, refer to the penultimate active round as round #1 and the last active round as round #2. Suppose  $i$  agents play in #1 and  $j$  in #2 (where  $i + j = l + 1$ .) For now, assume that  $i \geq 2$ . In all cases, consider the subgame (round #1) reached when all players are following the prescribed strategy, and condition on having reached this subgame.

First assume that  $i > 1$ . The expected payoff to an agent in round #1 is  $p_1 \cdot \frac{v(n-j)-v(n-j-i)}{i \cdot v(n-j)} + p_2 \cdot \frac{v(n-j)-v(n-j-i)}{i \cdot v(n)}$ , where  $p_1 = \Pr(\theta \leq \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n-j)-v(n-j-i)}{v(n)-v(n-j-i)}$  and  $p_2 = \Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n)-v(n-j)}{v(n)}$ . Consider a deviation to a later round by such an agent. By the inductive hypothesis, this would be a subgame with  $l$  or less players, and so all would play in the same round. Therefore, the expected payoff for such a deviation would be  $p_3 \cdot \frac{v(n)-v(n-j-1)}{(j+1)v(n)}$  and  $p_3 = \Pr(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n)-v(n-j-1)}{v(n)-v(n-j-i)}$ . The expected payoff to an agent in round #2 is  $p_4 \cdot \frac{v(n)-v(n-j)}{j \cdot v(n)}$ . Consider a deviation to round #1 by such an agent. By the inductive hypothesis, the remaining  $j-1 \leq l$  players would all play in the same round and so the expected payoff for a deviation would be  $p_4 \cdot \frac{v(n-j+1)-v(n-j-i)}{(i+1)v(n-j+1)} + p_5 \cdot \frac{v(n-j+1)-v(n-j-i)}{(i+1)v(n)}$ , where  $p_4 = \Pr(\theta \leq \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n-j+1)-v(n-j-i)}{v(n)-v(n-j-i)}$  and  $p_5 = \Pr(\theta > \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n)-v(n-j+1)}{v(n)-v(n-j-i)}$ . For an SPNE we require  $\frac{v(n-j)-v(n-j-i)}{v(n)-v(n-j-i)} \cdot \frac{v(n-j)-v(n-j-i)}{i \cdot v(n-j)} + \frac{v(n)-v(n-j)}{v(n)-v(n-j-i)} \cdot \frac{v(n-j)-v(n-j-i)}{i \cdot v(n)} \geq \frac{(v(n)-v(n-j-1))^2}{(j+1)v(n)(v(n)-v(n-j-i))}$  and  $\frac{(v(n)-v(n-j))^2}{j \cdot v(n)(v(n)-v(n-j-i))} \geq \frac{v(n-j+1)-v(n-j-i)}{v(n)-v(n-j-i)} \cdot \frac{v(n-j+1)-v(n-j-i)}{(i+1)v(n-j+1)} + \frac{v(n)-v(n-j+1)}{v(n)-v(n-j-i)}$ . Therefore, for an SPNE, we know that  $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j-i)}{v(n-j)} - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j-i)}{v(n-j+1)} - \frac{v(n-j+1)}{v(n)}) \geq \frac{(v(n)-v(n-j-1))^2}{(j+1)v(n)} - \frac{(v(n)-v(n-j))^2}{j \cdot v(n)}$ . In order to establish a contradiction, we show that this equation can never hold.

First, we observe that for additive complements, we have  $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j-i)}{v(n-j)} - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j-i)}{v(n-j+1)} - \frac{v(n-j+1)}{v(n)}) < \frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j+1)}{v(n)})$ . For this, we need  $\frac{v(n-j+1)-v(n-j-i)}{(i+1)v(n-j+1)} < \frac{v(n-j)-v(n-j-i)}{i \cdot v(n-j)}$ . By additive complements, we require  $\frac{2(n-j+1)-i}{(n-j+1)(n-j+2)} < \frac{2(n-j)-i+1}{(n-j)(n-j+1)}$ , which is equivalent to  $\frac{i+2}{n-j+2} > \frac{i-1}{n-j}$ . Now, we have  $\frac{i+2}{n-j+2} \geq \frac{i}{n-j} > \frac{i-1}{n-j}$ , since  $i \leq n-j$ . Based on this, it is sufficient for a contradiction to establish  $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j+1)}{v(n)}) < \frac{(v(n)-v(n-j-1))^2}{(j+1)v(n)} - \frac{(v(n)-v(n-j))^2}{j \cdot v(n)}$ . This is equivalent to:  $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2v(n)-v(n-j)) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2v(n)-v(n-j+1)) < \frac{(v(n)-v(n-j-1))^2}{(j+1)} - \frac{(v(n)-v(n-j))^2}{j}$ .

Note that the following identities hold for additive complements:

$$\begin{aligned} \frac{v(n-j)-v(n-j-i)}{i} \cdot (2v(n)-v(n-j)) &= (2n-2j-i+1)(2n(n+1)-(n-j)(n-j+1)) \\ \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2v(n)-v(n-j+1)) &= (2n-2j-i+2)(2n(n+1)-(n-j+1)(n-j+2)) \\ \frac{(v(n)-v(n-j))^2}{j} &= j(2n-j+1)^2 \\ \frac{(v(n)-v(n-j-1))^2}{j+1} &= (j+1)(2n-j)^2 \end{aligned}$$

Therefore we write:  $\frac{(v(n)-v(n-j-1))^2}{(j+1)} - \frac{(v(n)-v(n-j))^2}{j} = (j+1)(2n-j)^2 - j(2n-j+1)^2 = 4(n-j)^2 - j(j+1)$ .

And we can write:  $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2v(n) - v(n-j)) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2v(n) - v(n-j+1)) = (2n-2j-i+1)(2n(n+1) - (n-j)(n-j+1)) - (2n-2j-i+2)(2n(n+1) - (n-j+1)(n-j+2)) = 2(2n-2j-i+1)(n-j+1) - 2n(n+1) + (n-j+1)(n-j+2)$ . Left to prove is

$$2(2n-2j-i+1)(n-j+1) - 2n(n+1) + (n-j+1)(n-j+2) < 4(n-j)^2 - j(j+1) \quad (1)$$

The LHS is maximized for  $i = 1$ , and substituting for this it suffices to show  $4(n-j)^2 + 4(n-j) + (n-j+1)(n-j+2) - 2n(n+1) < 4(n-j)^2 - j(j+1)$ , or in other words:  $j(j+1) + 4(n-j) + (n-j+1)(n-j+2) - 2n(n+1) < 0$ , which is equivalent to  $2(j-1)^2 - 2j(n+1) < n^2 - 5n$ . Note that the LHS is negative for all values of  $i$  and  $n$  since  $j \geq 1$ , and the RHS is non-negative for  $n \geq 5$ . Therefore we know that this equation holds for all  $n \geq 5$  and all  $i+j \leq n$ . When  $n = 4$  this equation becomes,  $2(j-1)^2 - 10j < -4$  or  $2j^2 - 14j + 6 < 0$ .  $2j^2 - 14j + 6$  is less than 0 for all  $1 \leq j \leq 3$ . When  $n = 3$ , this equation becomes  $2j^2 - 12j + 8 < 0$ .  $2j^2 - 12j + 8$  is less than 0 for all  $1 \leq j \leq 2$ . Therefore we have established the desired result.

Finally, when  $i = 1$ , Lemma 7.3 tells us that this cannot be a SPNE.  $\square$

**Lemma 7.3.** *No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and  $l$  players in the last active round can be a SPNE with the proportional share rule for any additive complements valuation function, given that no strategy profile is a SPNE when there are either (a) at most  $l$  players in the last two active rounds, or (b) exactly  $l+1$  players in the last two active rounds with at least two players in the penultimate active round.*

*Proof.* Let  $i$  be the player who participates in the penultimate active round. We will establish this via strong induction on the number of rounds before  $T$  where agent  $i$  plays. The expected payoff is conditioned throughout on reaching the penultimate active round.

*Base Case:* No strategy profile with at least two active rounds in equilibrium in which there is one player in the penultimate active round and this round is in period  $T-1$  can be a SPNE. Let  $j$  denote the number of players in the last active round. Because we are focused on rounds  $T-1$  and  $T$  then the same analysis as was used for the  $i > 1$  case in the proof of Lemma 7.2 is valid here. Either the agent in the penultimate active round can usefully deviate later (in which case it necessarily plays in the same round as the other players since it plays in round  $T$ ), or an agent in round  $T$  can usefully deviate and play in round  $T-1$  with the singleton agent. For this, it is sufficient to note that the proof of Lemma 7.2 establishes that:  $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j-i)}{v(n-j)} - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j-i)}{v(n-j+1)} - \frac{v(n-j+1)}{v(n)}) < \frac{(v(n)-v(n-j-1))^2}{(j+1)v(n)} - \frac{(v(n)-v(n-j))^2}{jv(n)}$  for all  $i, j, n$  such that  $i, j \geq 1$  and  $i+j \leq n$ .

*Inductive Hypothesis:* No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and this round is  $r+1$  periods before  $T$  can be a SPNE. Consider what happens when the player in the penultimate active round deviates and goes later. By assumption (a) in the statement of the lemma we know that there can be at most two active rounds in the resulting subgame, because otherwise the last two active rounds would include  $l$  or less players. Then, by the inductive case for  $r$  periods to go, and by assumption (b) in the statement of the lemma, the only SPNE in the subgame following the deviation to a later round involves all players playing in the same round. Then, by the same analysis as for the base case, either the player in the penultimate active round can profitably deviate later or a player in the last active round can profitably deviate to the penultimate active round.  $\square$