Designing Incentives for Online Question-and-Answer Forums

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Abstract

We provide a game-theoretic model of sequential information aggregation motivated by online question-and-answer forums. An asker posts a question and each user decides when to aggregate a unique piece of information with existing information. When the quality exceeds a certain threshold, the asker closes the question and allocates points to users. We consider the effect of different rules for allocating points on the equilibrium behavior. A best-answer rule provides a unique, efficient equilibrium in which all users respond in the first round, for substitutes valuations over information. However, the best-answer rule isolates the least efficient equilibrium for complements valuations. We demonstrate alternate scoring rules that provide an efficient equilibrium for distinct subclasses of complements valuations, and retain an efficient equilibrium for substitutes valuations. We introduce a reasonable set of axioms, and establish that no rule satisfying these axioms can achieve the efficient outcome in a unique equilibrium for all valuations.

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1. Introduction

In online question-and-answer forums such as Yahoo! Answers, users can post questions and answer questions on wide variety of topics. In particular, Yahoo! Answers has 25 categories ranging from *Computers & Internet* to *Travel* to *Family & Relationships* to *Health*. Users may post discussion questions, factual questions or polls. In Yahoo! Answers, people do not exchange money for the exchange of information, but instead receive points for contributions that influence leaderboard and top-contributor designations, while also allowing users to post their own questions.[1]

We study a game-theoretic model of a problem of sequential information acquisition that is motivated by these online question-and-answer forums. In particular, our model is suitable for the study of the design of methods to assign scores to answers to factual questions, such as "What were the main causes of the Great Plague of London?" This is because we model the value of answers submitted to a question as strictly increasing over time, either because users incorporate their own private information with the information provided by earlier reports, or because the asker is able to piece together different responses and thus has an increasing value for the answers. Harper et al. Harper et al. (2009) have demonstrated that factual questions have a higher archival value than discussion questions, justifying the focus of this work. An example of a discussion question is, "What is your favorite movie of all time?" Our model is less suitable for these questions, where it makes less sense for subsequent answers to improve on earlier answers and to incorporate earlier responses.

Whereas the established model of *contest design* Moldovanu and Sela (2001, 2006) considers agents with costly but independent effort and seeks to maximize the total effort exerted across all agents, in our model

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[1]A user is placed into one of seven levels based on her number of points. The higher the level, the greater privileges a user will get in terms of the number of questions she can ask per day. All users have a profile where the number of points the user has, the level, and the percentage of best answers is clearly displayed. In addition to the point system, the "top contributor" is displayed at the top of the page for each semantic category and sub-category, and there is a leaderboard of the top ten users.

each agent can build on the work already contributed by other agents, and submit a solution that dominates all solutions so far submitted. In keeping the model simple, we choose not to model the cost of contributing an answer and model the asker as satisficing, with a private *quality threshold* at which she will close the question. In the model, the asker privately draws this quality threshold at random, and is satisfied with any answer with value above this threshold. The asker prefers to receive a satisfactory answer sooner rather than later, and closes the question as soon as the threshold is exceeded.

Each user holds a unique piece of information that is relevant to a question, and the strategic decision is decide when to report this information and aggregate it with previous reports. As information is reported it is aggregated into the responses, so that the value to the asker monotonically improves while the question remains open. In the case that multiple pieces of information are simultaneously revealed, we assume that the asker is able to aggregate the information, and we associate each user that contributed in the round with an answer equal in quality to that achieved through this aggregation. We study two distinct models for the way in which individual contributions contribute to an overall value of the asker. In particular, we study a *complements case*, in which each successive piece of information is worth more to the asker than the previous one, and a *substitutes case* in which each successive piece of information is worth less than the previous one.

Our interest is in characterizing the properties of subgame perfect Nash equilibria of this game under different rules for assigning points, and for users who seek to maximize their expected number of points. By delaying a contribution, a user runs the risk that the asker will be satisfied with the current answer and close the question. On the other hand, by delaying a contribution, a user can take advantage of contributions by other users and submit a better answer, thereby increasing the probability that a user's answer (if submitted before the question has been closed) will cross the quality threshold of the asker.

As a designer, we seek to understand which rules induce equilibria in which all users choose to contribute their respective pieces of information in the first round, and thus immediately and without delay. We first analyze the equilibria for a *best-answer* rule, which models the current Yahoo! Answers environment. We find that this rule is effective for substitutes valuations, where it isolates a subgame perfect Nash equilibrium in which all users reveal information in the first round. This is the efficient outcome, with the asker receiving a satisfactory answer as soon as possible for all possible quality thresholds. On the other hand, the best-answer rule is ineffective for complements information, where it isolates an equilibrium in which every user posts information in the very last round. For the case of complements information, the expected gain from an answer with higher quality, that comes from playing later and combining an answer with previous answers, is greater than the negative effect of delaying and risking that the question will close before submitting an answer.

In addressing this problem, we consider two alternative rules for assigning points to answers. The first rule is an *approval-voting rule*, parameterized by integer $k > 1$, in which the asker assigns one point to the most recent $k > 1$ answers (or some random k subset if more than k answers were received in the most recent round) upon closing the question. The approval-voting rule retains the efficient outcome in an equilibrium of the game for substitutes valuations. The approval-voting rule also enables the most efficient, all-going-first outcome in an equilibrium for complements information, under certain restrictions on the valuation function. But, the approval-voting rule also retains an equilibrium for complements information in which every user plays in the last round. More problematically, the approval-voting rule also introduces this inefficient outcome in an equilibrium of the game with substitutes valuations.

The second rule that we introduce is the *proportional-share* rule, in which the asker assigns a share of the total available points in proportion to the marginal value contributed by a user in the round in which the user participates. Like the approval-voting rule, the proportional-share rule enables the most efficient outcome outcome in equilibrium for a large class of complements information. In addition, the proportional-share rule retains the efficient outcome as an equilibrium for substitutes valuations, and unlike the approval-voting rule, this remains the unique equilibrium.

The approval-voting and proportional-share rules both avoid the incentives for delaying to the last round in the setting of complements information by spreading the score across more users. They do this in different ways. In particular, both the approval-voting and proportional-share rules are able to achieve the efficient outcome as a subgame perfect Nash equilibrium for certain classes of complements valuations. The approval-voting voting rule, but not the proportional-share rule, also introduces the inefficient outcome in an equilibrium for environments with substitutes valuations. On the other hand, the approval-voting rule is a simple generalization of the best-answer rule and likely more relevant to practice because it does not require new information from the asker when a question has been closed.

A natural question is whether there can be method of assigning points that is first best, in that it isolates the efficient outcome as a unique equilibrium for all possible complements and substitutes valuation functions. We obtain a negative result in this regard— we introduce three axioms, *anonymity, monotonicity* and *time-invariance*, and show that there is no payment rule that satisfies these properties and enables the efficient outcome in a unique subgame perfect Nash equilibrium.

1.1. Related Work

We believe this to be the first work studying online question-and-answer forums in a game-theoretic light. In terms of game-theoretic analysis of other systems of human computation (von Ahn and Dabbish, 2008), prior work has presented a game-theoretic analysis of the *ESP game* (Jain and Parkes, 2012) and the *PhotoSlap game* (Ho et al., 2007). These are so-called *Games with a Purpose*, games that are designed to be fun to play, with the added benefit that users are doing useful work in the process. While the game-theoretic analysis provided is specific to these games, these systems are similar to question-and-answer forums in that users are motivated by an artificial points system.

Closely related to the design of question-and-answer forums is the area of contests and all-pay auctions. Contests are situations in which multiple agents exert effort in order to win a prize. All agents bear the "cost" of the effort exerted regardless of whether they win a prize. Most of the literature on contest design has focused on the case where agents compete in a single contest for a unique prize and under a model of complete information (Dasgupta, 1986; Tullock, 1980; Varian, 1980). For instance, (Moldovanu and Sela, 2001) seek to understand how many prizes should be awarded, and of what value, where the principal has fixed resources, in order to maximize the total effort exerted across all agents. Still, the agents do not build from each other's solutions as in our model. A related model retains simultaneous submissions within a round, but considers the effect of subdividing the users into a set of parallel sub-contests, with the winners of these sub-contests competing in a subsequent round (Moldovanu and Sela, 2006).

A number of papers study crowdsourcing contests specifically (Archak and Sundararajan, 2009; Chawla, Hartline and Sivan, 2011; DiPalantino and Vojnovic, 2009). For example, DiPalantino and Vojnovic (2009) model a market with multiple contests, characterizing the equilibrium behavior of their model when workers decide on which contests to invest effort. Other work has adopted an all-pay auction analysis approach to address the principal's problem, e.g., how many prizes should be awarded, and of what value (Archak and Sundararajan, 2009; Chawla et al., 2011). Chawla et al. (2011) make the connection between crowdsourcing contests and optimal auction design, finding that the optimal crowdsourcing contest is a virtual valuation maximizer.

Cavallo and Jain (2012a,b) present an alternative model of crowdsourcing, where output is now a *stochastic* function of skill and effort rather than a *deterministic* function, design efficient (social welfare maximizing) mechanisms (Cavallo and Jain, 2012a), and compare these efficient mechanisms to winner-take-all mechanisms (Cavallo and Jain, 2012b). In (Archak and Sundararajan, 2009; Cavallo and Jain, 2012a,b; Chawla et al., 2011; DiPalantino and Vojnovic, 2009), the principal experiences disutility for the payments made and therefore apply to settings with real money. Other work considers models of user-generated content, where users contribute content of varying quality that is rated by other users and study which mechanisms (with attention-based rewards), induce high quality contributions from users (Ghosh and Hummel, 2011; Ghosh and McAfee, 2011, 2012). Ghosh and Hummel (2012) consider a setting with virtual points and identify a mechanism that can implement the principal-optimal outcome for a large class of utility functions (i.e. any utility function that is a linear combination of the goods submitted). The principal in their model does not experience disutility for the prize awarded and therefore, the model only applies for a setting with virtual points.

Most of this prior work adopts a model of production with quality that depends on the effort that is exerted, and the effort exerted in turn depends on the payment rule adopted in the design. In comparison, we model the strategic aspect facing a worker in a setting in which its contribution will be aggregated with contributions from other workers, and a user's strategic decision is in regard to when to contribute, rather than how much effort to exert. A limited number of previous models have been proposed of *sequential contests*, where users arrive at distinct time steps and decide how much effort to exert with the goal of winning a prize (Konrad and Leininger, 2007; Liu et al., 2011; Segev and Sela, 2011). Such work is different from ours, in that each agent is present only for a single, unique time step, whereas in our model agents are present for all time steps. Therefore, the strategic decision facing users is, how much effort to exert given the effort levels of the previous users (and the expected effort levels of future users), whereas in our model, the strategic decision facing users is when to participate.

There have been a number of empirical studies of online question-and-answer forums. For example, Nam et al. (2009) show that points are a factor in motivating users to participate in points-based question-and-answer forums. They study the Naver Knowledge-iN (KiN) system, the largest question and answer community in South Korea. They give a survey to twenty-six users of KiN and find that points are source of motivation for users, along with altruism, promoting personal businesses, learning and maintaining personal knowledge. Adamic et al. (2008) study contribution patterns in Yahoo! Answers and try to determine the extent to which certain statistical features such as response length effect the probability of an answer being chosen as a "best answer". Yang et al. (2008) conduct an empirical study of user behavior on *Taskcn*, a popular web-based knowledge sharing market in China (also known as a *Witkey*). These authors study user behavior over time, and find that a very small core of successful users manage to increase their win percentage over time and these users account for 20% of the winning answers. Strategic behavior is demonstrated, in that users learn over time to select tasks where they are competing against few opponents (and thus increase their chances of winning), and to select tasks with higher expected rewards.

2. Our Model

We focus on modeling how users participate in answering a single question posted by the asker. Each user $j \in \{1, \ldots, n\}$ has a unique piece of information I_j, with $\mathcal{I} = \{I_1, I_2, ..., I_n\}$. Even though information is private, the fact that everyone possesses a piece of information out of n total pieces is common knowledge. The information aggregation process proceeds over a set of discrete rounds, with at most $T > 1$ rounds, and closes in an earlier round if the value of the aggregate information reported up to and including that round exceeds a value threshold of the asker. Each user j observes the participation of other users and selects a single round in which to participate and aggregate I_j with the information reported so far.

Each piece of information is distinct but equivalent in terms of the value it provides to the asker, so that the asker's valuation function $v : \{0, 1, \ldots, n\} \to \mathbb{R}_{>0}$ depends only on the number of distinct pieces of information reported. We assume that $v(0) = 0$ and $v(j + 1) > v(j)$ for all $0 \leq j < n$. In posting an answer as the only user active in a round, the value of the answer is $v(\ell + 1)$ where ℓ users had previously posted an answer. By this, each user aggregates all previous answers in the processing of revealing his information. In posting an answer as one of a set of m users in a round, with ℓ previous posts, the value of the answer of each of these users is $v(\ell + m)$ to model the ability of the asker to aggregate all the information in this case of simultaneous revelation.

Depending on the nature of the question, the pieces of information related to the question may be complements or substitutes. For example, suppose the asker posts the question: "What should I do for a one-day visit to Boston?" The two pieces of information, "walk along the Freedom Trial" and "have lunch at Quincy Market (which is on the Freedom Trial)" are complements, because the value of knowing both pieces of information for the asker is higher than the sum of the values of only knowing a single piece of information. However, if the asker posts the question: "Where should I have lunch in Times Square?", the answers "Becco" and "Kodama" are substitutes for the asker, since the asker must choose between the two.

Let $\delta_j = v(j) - v(j-1)$ for $1 \leq j \leq n$.

Definition 2.1. In the *complements* case, the asker's valuation function must satisfy $\delta_j < \delta_{j+1}$ for all $1 \leq j < n$.

Definition 2.2. In the *substitutes* case, the asker's valuation function must satisfy $\delta_j > \delta_{j+1}$ for all $1 \leq j < n$.

The asker has a private *value threshold* θ, sampled uniformly on $[0, v(n)]$. The distribution from which θ is sampled is common knowledge. The asker prefers a satisficing answer (with value at least θ) as soon as possible, and closes the question in the first round in which the value of an answer meets or exceeds the threshold. Upon closing the question the asker adopts a rule by which it assigns points (any non-negative value, in general) to some subset of the users who have responded. Based on this, each user seeks to maximize her expected score. Because we choose not to associate a cost with the participation of a user, it is without loss to consider only strategies in which a user submits an answer in some round.

Let $b(t) \in \{0, \ldots, n\}$ denote the number of pieces of information revealed up to and including round $t \in \{0, 1, \ldots, T\}$.

Lemma 2.3. *The probability of stopping in round k, for $k \geq 1$, is $P(k) = \frac{(v(b(k))-v(b(k-1)))}{v(n)}$.*

Proof. The probability of stopping in round k is the probability that $\theta \leq v(b(k))$ and $\theta > v(b(k-1))$, which is just $\frac{(v(b(k))-v(b(k-1)))}{v(n)}$ for the uniform distribution. □

2.1. The Point Allocation Rules

In this paper, we examine the equilibrium behavior of users in the question-and-answer game under three different rules.

Best-Answer Rule. The best-answer rule models the method of assigning points currently used by Yahoo! Answers. In Yahoo! Answers, upon closing the question, the asker can select one answer as the best answer and the associated user is then awarded some fixed number of points. Without loss of generality, we normalize the number of points awarded to 1.

When the asker closes the question because the value has reached the threshold, the asker awards a single point to the user $i \in A$ that maximizes $b(t_i)$, where A is the set of agents who participated before the question closed. In other words, the best-answer rule awards a single point to the user who provides the answer with the largest total value. If there is a tie for the answer with the largest total value, ties are broken uniformly at random. Given our assumption that users incorporate information from previous rounds into their answers, the best-answer rule awards a point to the user who played in the last round before the question closed.

Approval-Voting Rule. Under the approval-voting scheme, the asker can provide the same reward to each of $k > 1$ users, where $k < n$. The number of winners, k is a design parameter. Note that if $k = 1$, this reduces to the best-answer rule.[2] Under the approval-voting rule, the k users who provide the answers with the largest total values are rewarded. Given our assumption that users incorporate the information from previous rounds into their answers, the approval-voting rule awards a point to the k most recent users.

In our model, we assume that the asker will always assign $k < n$ winners if possible and the k most recent users to answer before the question is closed will each receive a point, with ties broken uniformly at random. In the special case in which the question is closed and less than k users have responded, these users each receive one point. When more than k users respond in the most recent round, then a subset of k winners is selected uniformly at random. Similarly, when less than k users (say k_1) respond in the most recent round but more than $k - k_1$ users responded in the previous round, then a subset of $k - k_1$ users from the previous round are selected as winners uniformly at random.

Proportional-Share Rule. In the proportional-share rule, the asker is given a fixed number of points that she can distribute. Without loss of generality, we normalize the total number of points to distribute to 1. We assume that the asker distributes the point according to her valuation function. More specifically, suppose the question closes after $\ell \leq T$ active rounds and at each active round $t \leq \ell$ there are n_t participants. In the proportional-share rule, the asker distributes $\frac{v(b(1))}{v(b(\ell))}$ equally among the n_1 users participated in round 1, and, similarly, distributes $\frac{v(b(t))-v(b(t-1))}{v(b(\ell))}$ to the n_t users that participated in active round $t > 1$, where $v(b(t))$ denotes the value of the items received at the end of round t.

Given that our game is one of multiple time periods and that users have information about the game play in previous time periods, we use the subgame perfect Nash equilibrium concept to analyze this game. We use the notion of an *active round* in our analysis. A round is considered active if at least one user participates in that round, in equilibrium, when $\theta = 1$.

[2] It should be noted that Naver Knowledge-iN allows askers to select more than one best answer.

3. Equilibrium Analysis

3.1. Best-Answer Rule

Let h^t denote the history of play up until round t, that is, let h^t denote a complete specification of the agents that have submitted an answer up to but not including round t and the round in which they participated. Consider the following strategy for agent i:

$$s_i(h^t) = \begin{cases} \text{play} & \text{, if } t = T \\ \text{wait} & \text{, if } t < T \end{cases}$$

We show that all players adopting this strategy is a unique subgame perfect Nash equilibrium (SPNE) for the best-answer rule and with complements valuations.

Theorem 3.1. *For any valuation function satisfying the complements condition, the unique subgame perfect Nash equilibrium under the best-answer rule is the strategy profile in which all players always play in the last round.*

Proof. Consider an arbitrary subgame characterized by history h^t in which user i has not yet played. Assume at the start of this arbitrary subgame that m users have played thus far. Consider an arbitrary strategy s' of users $\neq i$. Let j denote the number of other users that play in the last round when i follows the prescribed strategy and plays in the last round. The expected payoff to agent i is $\Pr(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(m)}{v(n)}) \cdot \frac{1}{j+1}$. Now consider a single deviation by user i in period t, where ℓ denotes the number of other users that participate in round t under s'. We necessarily have $\ell \leq n - j - 1$. The expected payoff to agent i for *play* in this round is $\Pr(\frac{v(m)}{v(n)} < \theta \leq \frac{v(m+\ell+1)}{v(n)} | \theta > \frac{v(m)}{v(n)}) \cdot \frac{1}{\ell+1}$. Note that $\Pr(\theta > \frac{v(n-j-1)}{v(n)}) \cdot \frac{1}{j+1} = (\delta_{n-j} + ... + \delta_n) \cdot \frac{1}{j+1}$ and $\Pr(\frac{v(m)}{v(n)} < \theta \leq \frac{v(m+\ell+1)}{v(n)}) = (\delta_{m+1} + ... + \delta_{m+\ell+1}) \cdot \frac{1}{\ell+1}$. Since the valuation function satisfies the complements condition, we know that $\delta_i < \delta_j$ for all $i < j$. Therefore, we know that the mean value of the set $\{\delta_{n-j}, ..., \delta_n\}$ is strictly greater than the mean value of $\{\delta_{m+1}, ..., \delta_{m+\ell+1}\}$, so $\Pr(\theta > \frac{v(n-j-1)}{v(n)}) \cdot \frac{1}{j+1} > \Pr(\frac{v(m)}{v(n)} < \theta \leq \frac{v(m+\ell+1)}{v(n)}) \cdot \frac{1}{\ell+1}$ for any positive integer values of j and ℓ. Therefore we have $\Pr(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(m)}{v(n)}) \cdot \frac{1}{j+1} > \Pr(\frac{v(m)}{v(n)} < \theta \leq \frac{v(m+\ell+1)}{v(n)} | \theta > \frac{v(m)}{v(n)}) \cdot \frac{1}{\ell+1}$ for any positive integer values of j and ℓ.

This establishes that whatever the strategy of other players forward from any subgame, the strict best response of player i is to follow the prescribed strategy. This establishes that playing in the last round is a unique SPNE and completes the proof. ☐

Now consider the following strategy for player i:

$$s_i(h^t) = \begin{cases} \text{play} & \text{if } t = 1 \\ \text{no available action} & \text{if } t > 1 \end{cases}$$

Theorem 3.2. *For any valuation function satisfying the substitutes condition, the unique subgame perfect Nash equilibrium under the best-answer rule is the strategy profile in which all users play in the first round.*

Proof. Consider the first round. Fix an arbitrary strategy s' of users $\neq i$ and let j denote the number of other users that play in round $t = 1$. The expected payoff to agent i under the prescribed strategy is $\frac{v(j+1)}{v(n)} \cdot \frac{1}{j+1}$. Now consider a single deviation by user i in period 1, where now instead of playing in round 1, player i participates in a later round. Suppose that ℓ other users participate in this round (with $\ell \leq n-j-1$ necessarily, under s'.) Furthermore assume that m users have participated before this round, under s' (where $m \geq j$). The expected payoff to agent i is $\frac{p'}{\ell+1}$ where p' is the probability the threshold is first reached in this later round, conditioned on s'. In particular, $p' \leq \frac{v(\ell+1+m)-v(m)}{v(n)}$. Because of the substitutes assumption, $\frac{v(j+1)-v(0)}{v(n)} \cdot \frac{1}{j+1} > \frac{v(\ell+1+m)-v(m)}{v(n)} \cdot \frac{1}{\ell+1}$, for any values of j, l, and m, and the payoff to player i is always greater if she plays in the first round.

This establishes that it is a strict best response for agent i to play in the first round whatever the strategies of the other players, and establishes this as the unique SPNE. ☐

3.2. Approval-voting Rule

The approval-voting rule is parameterized by $k \in \{2, \ldots, n-1\}$, where k are the number of points awarded in total. In considering the case of complementary valuations, we first establish a useful characterization result. The proof of this lemma is deferred to the Appendix and it is obtained via strong induction on the number of agents in the last two active rounds.

Lemma 3.3. *All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the approval-voting rule (with $k > 1$) for any complements valuations.*

Consider the following partially-specified strategy for agent i:

$$s_i^\ell(h^t) = \begin{cases} \text{play} & \text{if } t = \ell \text{ and no one else has played} \\ \text{wait} & \text{if } t < \ell \text{ and no one else has played} \end{cases}$$

We first show that all players adopting this strategy profile is a subgame perfect Nash equilibrium for any value of $\ell \in \{1, ..., T\}$ for complements valuations that are not too complementary.

Theorem 3.4. *For any valuation function that satisfies the complements condition and $\frac{v(n)-v(n-1)}{v(n)} \leq \frac{k}{n}$, all users playing s_i^ℓ, for any value of ℓ, is a subgame perfect Nash equilibrium under the approval-voting rule for $k > 1$ winners.*

Proof. If all users i play strategy s_i^ℓ, the on-the-path behavior is for all users to play in round ℓ and the expected payoff to each player is $\frac{k}{n}$. Suppose that a player j deviates and goes earlier, we know from Lemma 3.3, that the remaining $n-1$ players will all play in the same round in any strategy profile that is a SPNE. Therefore, the expected payoff to a user j who deviates and goes earlier is less than $\frac{1}{n}$ in the complements case. Thus this deviation is not profitable.

Now suppose that a user j deviates by going later. By deviating to a later round, the expected payoff is $\frac{v(n)-v(n-1)}{v(n)}$, which is at most $\frac{k}{n}$ by assumption. This completes the proof, and establishes that all players following s_i^ℓ is a SPNE for any ℓ. □

Theorem 3.5. *For any valuation function that satisfies the complements condition and $\frac{v(n)-v(n-1)}{v(n)} > \frac{k}{n}$, the unique subgame perfect Nash equilibrium under the approval-voting rule is for all users to play last, in the on-the-path play.*

Proof. Given Lemma 3.3, it suffices to consider strategy profiles in which all players participate in the same round in every subgame. First suppose that all players play in round $\ell < T$ in equilibrium. The payoff to any player is $\frac{k}{n}$. Deviation to a later round obtains $\frac{v(n)-v(n-1)}{v(n)} > \frac{k}{n}$, and is profitable. Consider $\ell = T$. Suppose player j deviates to an earlier round. Lemma 3.3 tells us that all remaining players will participate in the same round. Therefore the expected payoff to j is $\frac{v(1)}{v(n)} < \frac{1}{n} < \frac{k}{n}$, where the first inequality is from the complements property. Thus, the unique subgame perfect Nash equilibrium is for all users to play last. □

Thus we know for the case of complements valuations, if the valuation function is "very complementary," the equilibrium analysis provides the same outcome as with the best-answer rule, but for less extreme valuations, we have pooling equilibria, where all users playing in the same round, for any round, is an equilibrium.

Now we shift attention to the case of substitutes valuations. Similar to the complements case, we get a complete characterization result for the case of substitutes valuations. This result is established via strong induction on the number of agents in the last two active rounds. The proof is deferred to the Appendix.

Lemma 3.6. *All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the approval-voting rule (with $k > 1$) for any substitutes valuations.*

Again consider the partially-specified strategy profile $s_i^\ell(h^t)$ for some $\ell \in \{1, \ldots, T\}$. The following observation is for valuations that are not too substitutable.

Theorem 3.7. *For any valuation function that satisfies the substitutes condition and $\frac{v(1)}{v(n)} \leq \frac{k}{n}$, all users playing s_i^ℓ, for any value of ℓ, is a subgame perfect Nash equilibrium under the approval-voting rule for $k > 1$ winners.*

Proof. If all players play strategy s_i^ℓ the expected payoff to each player is $\frac{k}{n}$. Consider a player j who deviates and goes earlier. Lemma 3.6 tells us that the remaining $n-1$ players will play in the same round in any SPNE. Therefore, we know that the expected payoff of a user j is $\frac{v(1)}{v(n)} \cdot 1 < \frac{k}{n}$ by assumption. Thus this deviation is not profitable. Considering the other players, the expected payoff of such a player, conditioned on making it to the next round, for the stipulated strategy, is $\frac{k}{n-1}$. Now consider a deviation by user j to a later round. The expected payoff is $\frac{v(n)-v(n-1)}{v(n)} < \frac{1}{n} < \frac{k}{n}$, where the first inequality is by the substitutes property. This completes the proof, and establishes that all players following s_i^ℓ is a SPNE, for any ℓ. □

We now establish that the approval-voting rule isolates the most efficient outcome for "very substitutable" valuations.

Theorem 3.8. *For any valuation function that satisfies the substitutes condition and $\frac{v(1)}{v(n)} > \frac{k}{n}$, the unique subgame perfect Nash equilibrium under the approval-voting rule is for all users to play first, in the on-the-path play.*

Proof. Given Lemma 3.6, it suffices to consider strategy profiles in which all players participate in the same round in every subgame. First suppose all players play in round $\ell > 1$. The payoff to any player is $\frac{k}{n}$. The expected payoff to a player j who deviates to an earlier round is at least $\frac{v(1)}{v(n)}$, and so greater than $\frac{k}{n}$ by assumption. Consider $\ell = 1$. The expected payoff to a player who deviates to a later round is $\frac{v(n)-v(n-1)}{v(n)} < \frac{1}{n} < \frac{k}{n}$, where the first inequality follows by the substitutes property, and this is not profitable. This completes the proof. □

In other words, for the case of substitutes, we get that if the valuation function is very substitutable, the unique subgame perfect Nash equilibrium is for all players to go first. For other substitutes valuations, every pooling profile is a subgame perfect Nash equilibrium.

3.3. Proportional-Share Rule

The approval-voting rule can introduce the efficient, all play first outcome in an equilibrium for certain complements valuations, but also introduces an equilibrium in which all users play in the last round for certain substitutes valuations. To this end, we consider the proportional-share rule, and seek to understand whether it is possible to obtain the efficient outcome for complements valuations without introducing the least efficient outcome for substitutes valuations.

Similar to the approval-voting rule, we get a strong characterization result for the proportional-share rule. For this characterization result, we assume additional structure on complements valuations, namely additive complements. Our characterization result holds for all additive complements valuations, i.e. for all $c > 0$. Theorem 3.10 is established by strong induction on the number of players in the last two active rounds and the proof is deferred to the Appendix.

Definition 3.9. We say that a valuation function exhibits additive complements if and only if $v(1) = c$, $v(2) - v(1) = 2c$, $v(3) - v(2) = 3c$, ..., $v(n) - v(n-1) = nc$ for any $c > 0$. In other words, $v(1) = c$, $v(2) = 3c$, ..., $v(n-1) = \frac{n(n-1)c}{2}$, $v(n) = \frac{(n+1)nc}{2}$.

Theorem 3.10. *All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the proportional-share for any additive complements valuations.*

Thus for the case of additive complements, it suffices to consider cases where agents play in the same round in every subgame.

Lemma 3.11. *For any valuation under the proportional-share rule and any strategy profile in which all users playing in the same round: (a) if $\frac{v(1)}{v(n)} \leq 1 - \sqrt{\frac{n-1}{n}}$, a user cannot profitably deviate by going in an earlier round, and (b) if $\frac{v(n-1)}{v(n)} \geq 1 - \sqrt{\frac{1}{n}}$, a user cannot profitably deviate by going in a later round.*

Proof. Consider the strategy profile consisting of all users going in the same round. The expected payoff of each user is $\frac{1}{n}$. Consider a user who deviates by playing in a later round. The expected payoff of such a user is $(1-p)\cdot(1-p)$, where $p = \frac{v(n-1)}{v(n)}$. In order for this deviation not to be profitable, we need $(1-p)^2 \leq \frac{1}{n}$, or equivalently, $p \geq 1 - \sqrt{\frac{1}{n}}$. Now consider a user who deviates by playing in an earlier round. Theorem 3.10 tells us that the remaining $n-1$ players will play in the same round in an SPNE. Therefore, the expected payoff of such a user is $p + (1-p)\cdot p$, where $p = \frac{v(1)}{v(n)}$. In order for this deviation not to be profitable, we need $p + (1-p)\cdot p \leq \frac{1}{n}$, or equivalently, $p \leq 1 - \sqrt{\frac{n-1}{n}}$. □

In fact, for $n \geq 3$, we have pooling equilibria, where all users playing in the same round is an equilibrium, for any round. For the case of $n = 2$, the only equilibrium is in which both players play first.

Theorem 3.12. *For the case of additive complements, with $n \geq 3$, the set of SPNE involve all users playing in the same round for any round.*

Proof. From Theorem 3.10, it suffices to consider strategy profiles where all users play in the same round. From Lemma 3.11, it suffices to show that $\frac{v(1)}{v(n)} \leq 1 - \sqrt{\frac{n-1}{n}}$ and $\frac{v(n-1)}{v(n)} \geq 1 - \sqrt{\frac{1}{n}}$. Since $n(n+1)^2 < (n-1)(n+2)^2$ for all $n \geq 3$, we know that $\frac{n-1}{n} < (\frac{(n-1)(n+2)}{n(n+1)})^2$ and $\sqrt{\frac{n-1}{n}} < \frac{n(n+1)-2}{n(n+1)} = 1 - \frac{2}{n(n+1)}$, so $\frac{2}{n(n+1)} < 1 - \sqrt{\frac{n-1}{n}}$, and equivalently $\frac{v(1)}{v(n)} < 1 - \sqrt{\frac{n-1}{n}}$ for additive complements. Since $(n+1)^2 > 4n$ for all $n > 1$, we know that $\frac{1}{n} > (\frac{2}{n+1})^2$ and equivalently $\frac{1}{n} > (1 - \frac{v(n-1)}{v(n)})^2$ for additive complements, and so $\sqrt{\frac{1}{n}} > 1 - \frac{v(n-1}{v(n)}$ and $\frac{v(n-1)}{v(n)} > 1 - \sqrt{\frac{1}{n}}$. □

In fact, one can also see from Lemma 3.11 that when $\frac{v(1)}{v(n)} > 1 - \sqrt{\frac{n-1}{n}}$ and $\frac{v(n-1)}{v(n)} < 1 - \sqrt{\frac{1}{n}}$ hold there are no pooling SPNE and thus there must be separating equilibria for complements valuations that satisfy these conditions under the proportional-share rule. An example of valuation function that has a separating equilibrium under the proportional-share rule is as follows: $v(1)/v(3) = 0.19$, $v(2) - v(1)/v(3) = 0.21$, $v(3) - v(2)/v(3) = 0.6$.

Finally we show that the efficient "all-going-first" outcome is preserved in the unique equilibrium for substitutes valuations under the proportional-share rule.

Theorem 3.13. *For any valuation that satisfies the substitutes condition, the unique subgame perfect Nash equilibrium under the proportional-share rule is the strategy profile consisting of all users going first in every subgame.*

Proof. Consider the strategy profile in which all players play first. Lemma 3.11 tells us that if all users are going in the first round, no user has incentive to deviate if and only if the valuation function satisfies the condition: $\frac{v(n-1)}{v(n)} \geq 1 - \sqrt{\frac{1}{n}}$, which is always satisfied by any valuation function that satisfies the substitutes condition. Now consider the situation where all users go in the same round, that is not the first round. The expected payoff of a user who deviates by playing in an earlier round is *at least* $\frac{v(1)}{v(n)}$, regardless of the game play of other agents. We know that $\frac{v(1)}{v(n)} > \frac{1}{n}$, and thus a player always wants to deviate and go earlier. Consider any strategy profile in which there are two or more active rounds. Suppose that j users play in the last active round. The expected payoff of a user who participates in the last active round is $(1 - \frac{v(n-j)}{v(n)}) \cdot \frac{v(n)-v(n-j)}{jv(n)}$. Consider the expected payoff of a user in the last round who deviates by going in the first active round. Suppose that i users participate in the first active round, including the user who deviated. Her expected payoff is *at least* $\frac{v(i)}{v(n)} \cdot \frac{1}{i}$, regardless of the game play that follows. For any valuation function that satisfies the substitutes condition, we know that $\frac{v(i)}{i} \geq \frac{v(n)-v(n-j)}{j}$, so $\frac{v(i)}{iv(n)} > (1 - \frac{v(n-j)}{v(n)}) \cdot \frac{v(n)-v(n-j)}{jv(n)}$. Thus no strategy profile in which there are two or more active rounds can be a Nash equilibrium. □

3.4. Discussion

While the model makes a number of simplifying assumptions, the results can be generalized in some places. First, the results hold for a larger family of distributions (over the asker's threshold) than just the uniform distribution. They hold for any distribution that preserves the complements or substitutes nature of the valuation. More specifically, we had defined the $\delta_j = v(j) - v(j-1)$ for all $1 \leq j \leq n$ and θ was drawn uniformly from $[0, v(n)]$, but we could equivalently define the δ_j in terms of the probability distribution, e.g. $\delta_j = \Pr(v(j-1) < \theta \leq v(j))$. Defined this way, the results given for complements valuations would hold for any threshold distribution that satisfies the property $\Pr(v(j-1) < \theta \leq v(j)) < \Pr(v(j) < \theta \leq v(j+1))$ (or equivalently $\delta_j < \delta_{j+1}$). Similarly, the results given for substitutes valuations would hold for any threshold distribution that satisfies the property $\Pr(v(j-1) < \theta \leq v(j)) > \Pr(v(j) < \theta \leq v(j+1))$ (or $\delta_j > \delta_{j+1}$). In addition, our results also hold in an equivalent model where there is a threshold integer t drawn uniformly at random from $[0, n]$ and representing the contribution at which the asker's value stops increasing, such that $v(j+1) > v(j)$ for all $j < t$ and $v(j+1) = v(j)$ for all $j \geq t$.

Other assumptions that we make in the treatment are that (a) the number n of users (or participants) is common knowledge, (b) the users are all present at all time steps, and (c) the users know the structure of the asker's valuation function, and in particular whether it is a subclass of substitutes or complements for some of the analysis. In fact, none of our analysis depends in a critical way on users having knowledge of the number of users in the system. [3] Considering the analysis for the best-answer rule, the results extend to a setting with arrivals and departures. For substitutes valuations, a user will want to participate in the first round that she is present. For complements valuations, a user will want to participate in the last round that she is present. The results are also robust, in that they do not require specific knowledge of the asker's valuation beyond it being substitutes or complements.

Turning to the approval-voting rule, this analysis does require knowledge of a specific subclass of substitutes or complements valuations, and we do not see how to generalize the analysis to users who arrive and depart. Such a generalization would likely require additional conditions on the structure of the valuation function. The current results require conditions on the relative value of the first item for substitutes valuations as compared to k and n (and the last item for complements valuations). For arbitrary arrival and departure times, we would need conditions on the relative value of each item. For the proportional-share rule, specific knowledge of a subclass of valuations is required for the complements analysis but not for the substitutes analysis. For substitutes valuations, the equilibrium in which all users contribute information in the first round generalizes to a setting with arrival and departures, since the analysis involves a dominant strategy argument. On the other hand, it does not seem straightforward to generalize the analysis of proportional-share under complements to handle arrivals and departures.

4. An Axiomatic Treatment

The best-answer rule isolated an equilibrium that supports the efficient, all-going-first, outcome for substitutes valuations. But on the other hand, the unique equilibrium has all users waiting until the last possible round to reveal their information for the case of complements valuations. In comparison, both the approval-voting and proportional-share rules are successful in attaining the efficient outcome in an equilibrium for certain subclasses of complements valuations. By tuning parameter k, the approval-voting rule can enable this for a larger class of complements valuations than the proportional share rule. Still, the least efficient outcome is retained in the equilibrium for some complements valuations under both rules. Another consideration is that the approval-voting rule, but not the proportional-share rule, also introduces an equilibrium that corresponds with the least efficient outcome for substitutes valuations.

These results beg the question: *Does there exist a rule for assigning points that isolates the best possible equilibrium for all asker valuations?* To answer this question we introduce three reasonable axioms for rules for assigning points in the context of question-and-answer forums, and prove that no rule can meet these

[3] We note for the approval-voting scoring rule, that some of the results require knowing the relative value of the first item with respect to k and n (for substitutes) and the relative value of the last item with respect to k and n for complements, but that this is a weaker condition than knowing n exactly.

axioms and always isolate a unique equilibrium in which the outcome is efficient, for all substitutes and complements valuations.

In building some intuition, consider the following three possible rules:

1. "Pay you only if you go first": In this rule, the asker pays each user who goes in the first round $\frac{1}{j}$, where j is the number of users who participate in the first round, and pays everyone else 0.
2. "Second-to-last"[4]: In this rule, the asker pays each user who goes in the second to last active round $\frac{1}{j}$, where j is the number of users who participate in the round and pays everyone else 0. If there is only one active round, players each receive $\frac{1}{j}$, where j is the number of users who participate in that active round.
3. "Uniform": The asker pays each user $\frac{1}{j}$ regardless of which round she participates in, where j is the number of users who get their information in before the question closes.

In the case of the first and the third rules, playing first is a dominant strategy, and the only subgame perfect Nash equilibria. In the first rule, an agent's expected payoff of playing in the first round is strictly positive whereas it is zero for playing in a later round. In the third rule, we note that for a fixed value of θ, an agent is paid the same regardless of which round he plays, as long as she gets her answer in. Therefore, an agent will always want to play first.

In the second rule, the all-going-first outcome is supported in the subgame perfect Nash equilibrium, because any player that deviates and goes later will receive a payoff of 0, for any value of θ. However, this is not a unique equilibrium. For example, all users playing in round ℓ, for all $1 < \ell < n$, is also a subgame perfect Nash equilibrium. A deviation of playing later yields payoff of 0, as opposed to $\frac{1}{n}$. If a user deviates and participates in round $\ell' < \ell$, and the remaining $n-1$ players respond with $n-2$ players participating in round $\ell' + 1$, followed by the last player in round $\ell' + 2$, this is not a profitable deviation.

Let us now introduce three natural axioms for rules for assigning points to users in our model of sequential information revelation. For this, let a *configuration* $\vec{c} = (t_1, t_2, ..., t_n)$, denote a realization of play in which user i participates in round t_i, unless the question closes before that round. For example, the configuration corresponding to an equilibrium strategy profile defines the rounds in which each agent plays for $\theta = 1$. Let $\vec{p}(\vec{c}, \theta) = (p_1, p_2, ..., p_n)$ denote the expected payoff to each player for configuration $\vec{c}$ and threshold θ, as induced by a rule, where the expectation is taken with respect to any randomization within the rule. Let $b_{\text{first}}(\vec{c}, t)$ and $b_{\text{last}}(\vec{c}, t)$ denote the total number of answers submitted in configuration $\vec{c}$ after one of the agents (if any) plays in round t and at the end of round t, respectively. Adopt $v(b(t))$ as shorthand for $v(\vec{c}, b_{\text{last}}(t))$, and the total value that accrues by the end of round t.

1. **Anonymous**: A rule is anonymous if, for all asker valuation functions, and any configuration $\vec{c_1} = (t_1, t_2, ..., t_n)$, any threshold θ, and any permutation σ, where $\vec{c_2} = \sigma(\vec{c_1})$, we have that $\vec{p}(\vec{c_2}, \theta) = \sigma(\vec{p}(\vec{c_1}, \theta))$.
2. **Time-Invariance**: A rule is time invariant if, for all asker valuation functions, and any pair of configurations $\vec{c_1} = (t_1, t_2, ..., t_n)$, $\vec{c_2} = (s_1, s_2, ..., s_n)$ such that $s_i - t_i = d$ for all $1 \leq i \leq n$, for some integer d, then $p_i(\vec{c_1}, \theta) = p_i(\vec{c_2}, \theta)$ for all $1 \leq i \leq n$.
3. **Value-Monotonic**: A rule is value monotonic if there exists an $\alpha \in [0, 1]$ such that, for every asker valuation function, we have:
   (a) For every configuration, every pair of players i, j, and every $\theta > \{v(b(t_i - 1)), v(b(t_j - 1))\}$, then $v'_\alpha(\vec{c}, t_i) \geq v'_\alpha(\vec{c}, t_j) \ \Rightarrow \ p_i(\vec{c}, \theta) \geq p_j(\vec{c}, \theta)$, **and**
   (b) For every configuration $\vec{c}$, every pair of players $(i, j) \in \arg\max_{(i,j)}[v'_\alpha(\vec{c}, t_i) - v'_\alpha(\vec{c}, t_j)]$, there exists a $\theta > \{v(b(t_i - 1)), v(b(t_j - 1))\}$ for which $v'_\alpha(\vec{c}, t_i) > v'_\alpha(\vec{c}, t_j) \ \Rightarrow \ p_i(\vec{c}, \theta) > p_j(\vec{c}, \theta)$,

   where *contributed value* $v'_\alpha(\vec{c}, t)$ is defined to be either $v(b_{\text{last}}(\vec{c}, t)) - \alpha \cdot v(b_{\text{last}}(\vec{c}, t) - 1)$ **or** $v(b_{\text{first}}(\vec{c}, t)) - \alpha \cdot v(b_{\text{first}}(\vec{c}, t) - 1)$.

To give examples, if $\alpha = 1$ and $v'_\alpha(\vec{c}, t)$ is defined in terms of b_{first}, then value monotonicity insists on properties on the payoff to a player that depend on the marginal value contributed as though the player were to play first in the round in which it plays. On the other hand, if $\alpha = 0$ and $v'_\alpha(\vec{c}, t)$ is defined in terms of b_{last},

[4]We would like to acknowledge Yoav Wilf for suggesting this rule.

then value monotonicity insists on properties on the payoff to a player that depend on comparing the total value at the end of the round in which the player submits an answer. Essentially, the value-monotonicity axiom seeks to be agnostic about whether the appropriate "measure" of value contributed is marginal or cumulative ($\alpha = 1$ or 0) and whether a player is considered to go first in a round or last in a round.

Condition 3 (a) is a weak monotonicity property while condition 3 (b) insists on strong monotonicity for at least a pair of players (i, j) for which the difference on contribution is the greatest and for at least some threshold, θ. Note that 3 (b) holds vacuously for a configuration $\vec{c}$ in which all players play in the same round because contributed value $v'_\alpha(\vec{c}, t_i) = v'_\alpha(\vec{c}, t_j)$ for all i, j and any definition of v' and α. A rule is value monotonic if 3 (a) and (b) hold for some choice of α and some choice of v'_α, which can be alternatively constructed in terms of b_{first} or b_{last}. A rule is *not* value monotonic if there is no such α and v'_α combination under which it meets conditions 3 (a) and 3 (b) for all valuation functions and all configurations.

Remark 4.1. *The "pay-you-if-you-go-first" rule satisfies anonymity, but violates time-independence and value-monotonicity.*

Proof. The payoffs under this rule are indifferent to permutations of the configuration, however the payoffs are not indifferent to shifts in time. All users participating in round $\ell = 1$ will lead to a payoff of $\frac{1}{n}$ for each. All users participating in round $\ell > 1$ will lead to a payoff of 0 for each. Now we consider value-monotonicity. Consider $\theta = 1$ and the configuration $\vec{c}$ in which player i plays in the first round and all other players play in the second round. In this scenario, player i will receive a payoff of 1, while all other players receive a payoff of 0. However, for any complements valuation, $v(1) < v(j) - v(j-1) \leq v(j) - \alpha v(j-1)$ for any $\alpha \in [0, 1]$, any $j > 1$. From this, we have $v'_\alpha(\vec{c}, t_i) = v(1) - \alpha v(0) = v(1) < v(j) - \alpha v(j-1) \leq v'_\alpha(\vec{c}, t_r)$, for any player r to play in the second round, where the first equality and the final inequality both hold irrespective of whether v'_α is defined on b_{last} or b_{first}. ☐

Remark 4.2. *The "second-to-last" rule satisfies time-independence and anonymity, but violates value-monotonicity.*

Proof. The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time. Now we consider value-monotonicity. Consider $\theta = 1$ and the configuration c in which player i plays in the first round and all other players play in the second round. In this scenario, player i will receive a payoff of 1, while all other players receive a payoff of 0. The rest of the proof follows according to that of Remark 4.1. ☐

Remark 4.3. *The uniform rule satisfies anonymity and time-independence, but violates value-monotonicity.*

Proof. The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time. Now we consider value-monotonicity and specifically 3 (b). Consider a configuration $\vec{c}$ in which each player plays in a separate round. Consider any complements valuation function and players (i, j) maximizing the difference in contributed value. For any definition of adjusted value, we must have $v'_\alpha(\vec{c}, t_i) > v'_\alpha(\vec{c}, t_j)$ since the players play in different rounds; e.g., if $\alpha = 0$ and v'_α is defined on b_{last} then this is the total value in a round. But uniform assigns the same score to every player who answers before the question closes, and so there is no θ that allows both i and j to play for which i's payoff is higher than j's in configuration $\vec{c}$. ☐

We can revisit the best-answer, approval-voting and proportional-share rules introduced earlier, and consider them from this axiomatic perspective.

Remark 4.4. *The best-answer rule satisfies anonymity, time-independence, and value-monotonicity.*

Proof. The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time. Now we consider value-monotonicity for $\alpha = 0$ and v'_α defined on b_{last} so that $v'_\alpha(\vec{c}, t_i) = v(b(t_i))$. Now, consider any configuration $\vec{c}$, any i, j and restrict attention w.l.o.g. to θ for which both i and j play. For 3 (a), suppose $v'_\alpha(\vec{c}, t_i) \geq v'_\alpha(\vec{c}, t_j)$, so that we must have $t_i \geq t_j$. Then, we have $p_i(\vec{c}, \theta) \geq p_j(\vec{c}, \theta)$ since if $t_i > t_j$ then $p_j(\vec{c}, \theta) = 0$, while if $t_i = t_j$ then the (expected) payoff is the same. Finally, if $v'_\alpha(\vec{c}, t_i) > v'_\alpha(\vec{c}, t_j)$ and so $t_i > t_j$, we have $p_i(\vec{c}, \theta) > p_j(\vec{c}, \theta)$ for $\frac{v_i(b(t_i)-1)}{v(n)} < \theta \leq \frac{v_i(b(t_i))}{v(n)}$ because $p_j(\vec{c}, \theta)$ is 0 and agent i competes to win with the other players (if any) in t_i. ☐

Remark 4.5. *The approval-voting rule satisfies anonymity, time-independence, and value-monotonicity.*

Proof. The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time. Now we consider value-monotonicity for $\alpha = 0$ and v'_α defined on b_{last}, so that $v'_\alpha(\vec{c}, t_i) = v(b(t_i))$. Consider any configuration $\vec{c}$, any i, j and restrict attention w.l.o.g. to θ for which both i and j play. For 3 (a), suppose $v'_\alpha(\vec{c}, t_i) \geq v'_\alpha(\vec{c}, t_j)$, so that we must have $t_i \geq t_j$. Then, if $t_i > t_j$ then whenever j scores i also scores and thus $p_j(\vec{c}, \theta) \leq p_i(\vec{c}, \theta)$. If $t_i = t_j$ then their (expected) payoff is the same. Then, consider a pair (i, j) that maximizes the difference $v(b(t_i)) - v(b(t_j))$, so that i plays in the first active round and j in the last active round. 3 (b) holds trivially for a configuration in which all players play in the same round, and so consider the case where $t_i > t_j$. Fix $\theta = 1$. For all valuation functions, we have if j scores then i scores. Moreover, conditioned on i receiving points, j's expected payoff is strictly less than 1 because j must compete for any remaining points $\max(0, k - (n - n(t_i)))$ where $n(t_i)$ play in round t_i with $n(t_i)$ players, with $n(t_i) > \max(0, k - (n - n(t_i)))$ since $k < n$. This completes the proof. ☐

Remark 4.6. *The proportional-share rule satisfies anonymity, time-independence and value-monotonicity.*

Proof. The payoffs under this rule are indifferent to permutations of the configuration and indifferent to shifts in time. Now we consider value-monotonicity, with $\alpha = 1$ and v'_α defined on b_{first}, so that $v_\alpha(\vec{c}, t_i) = v(b(t_i - 1) - 1) - v(b(t_i - 1))$. Consider any configuration $\vec{c}$, any i, j and restrict attention w.l.o.g., to any θ for which both i and j play. First suppose that $v(b(t_i - 1) - 1) - v(b(t_i - 1)) = v(b(t_j - 1) - 1) - v(b(t_j - 1))$. For both substitutes and complements valuations, we must have $t_i = t_j$, from which the expected payoff is the same to i and j for any θ that allows round $t_i = t_j$. This establishes part of what is required for 3 (a).

Second, suppose that $v(b(t_i - 1) - 1) - v(b(t_i - 1)) > v(b(t_j - 1) - 1) - v(b(t_j - 1))$. We establish 3 (a) and 3 (b) by showing that for *all* θ in which both players play, i's payoff is strictly greater than j's. For complements valuations, we must have $t_i > t_j$. Conditioned on both agents playing, agent i's expected payoff is $\frac{v(b(t_i)) - v(b(t_i - 1))}{v(b(t_r))} \cdot \frac{1}{i}$, where the game terminates in round r and agent j's expected payoff is $\frac{v(b(t_j)) - v(b(t_j - 1))}{v(b(t_r))} \cdot \frac{1}{j}$. Then, we have $\frac{v(b(t_i)) - v(b(t_i - 1))}{v(b(t_r))} \cdot \frac{1}{i} > \frac{v(b(t_j)) - v(b(t_j - 1))}{v(b(t_r))} \cdot \frac{1}{j}$ by complements. For substitutes valuations, we must have $t_i < t_j$ and agent i's expected payoff is $\frac{v(b(t_i)) - v(b(t_i - 1))}{v(b(t_r))} \cdot \frac{1}{i}$, compared to $\frac{v(b(t_j)) - v(b(t_j - 1))}{v(b(t_r))} \cdot \frac{1}{j}$ for agent j. Then, we have $\frac{v(b(t_i)) - v(b(t_i - 1))}{v(b(t_r))} \cdot \frac{1}{i} > \frac{v(b(t_j)) - v(b(t_j - 1))}{v(b(t_r))} \cdot \frac{1}{j}$ by substitutes. ☐

The following theorem establishes that we cannot have a rule satisfy all three axioms and isolate the most efficient, all-going-first equilibrium as the unique subgame perfect Nash equilibrium.

Theorem 4.7. *There is no rule for assigning points to answers that satisfies anonymity, time-independence and value monotonicity and isolates the all-going first equilibrium as the unique subgame perfect Nash equilibrium for all asker valuation functions.*

Proof. Assume otherwise, that is, assume that there exists a rule that satisfies anonymity, time-independence and value monotonicity and isolates the all-going first equilibrium as the unique subgame perfect Nash equilibrium for all asker valuation functions. This means that if players all go in the first round, there is no profitable deviation (of going later). Now consider the strategy profile in which players all play in the t^{th} round where $t > 1$. First observe that there is no useful deviation by a player going later, because by time-independence, this profitable deviation would still be available for the $t = 1$ strategy profile. Hence it suffices to consider deviations (say, by player 1) to an earlier round $\ell < t$. The payoff to player 1 from playing in round t is $\frac{1}{n}$ by anonymity.

Consider a complements valuation, and let $\vec{c}$ denote the configuration where 1 plays in round ℓ and arbitrary play by the rest in subsequent rounds. By complements, we have $v(1) - \alpha v(0) = v(1) < v(m) - v(m-1) \leq v(m) - \alpha v(m-1)$ for all $\alpha \in [0, 1]$, any $m > 1$. From this, then for *any* construction of contributed value v'_α, i.e. any $\alpha \in [0, 1]$ and with the use of b_{first} or b_{last}, we must have $v'_\alpha(\vec{c}, \ell) < v'_\alpha(\vec{c}, t_j)$ for any $j \neq 1$. For the moment, consider $\theta > \frac{v(n-1)}{v(n)}$, i.e. high enough so that all players play. By value monotonicity, we must have $p_1(\vec{c}, \theta) \leq p_j(\vec{c}, \theta)$ for all $j \neq 1$.

Denote by n an agent who plays in the last active round in $\vec{c}$. By complements, pair $(1, n)$ maximizes the difference $v'_\alpha(\vec{c}, t_i) - v'_\alpha(\vec{c}, t_j)$ for any construction of the contributed value function v'_α. To see this: if $\alpha = 0$ then $v'_\alpha(\vec{c}, t_j)$ is minimized for $j = 1$, for either $v'_\alpha(\vec{c}, t_j) = v(b_{\text{first}}(\vec{c}, t_j))$ or $v'_\alpha(\vec{c}, t_j) = v(b_{\text{last}}(\vec{c}, t_j))$.

Similarly, $v'_\alpha(\vec{c}, t_j)$ is maximized for an agent that plays in the last active round. Alternatively, if $\alpha = 1$ then $v'_\alpha(\vec{c}, t)$ is the marginal value contributed, either by the first answer or the last answer in round t, and by complements this is minimized for the first active round and maximized for the last active round. Clearly, $(1, n)$ is also a maximizing pair for any $\alpha \in (0, 1)$. Then, by strict value monotonicity (3 b), there must exist a $\theta > \frac{v(n-1)}{v(n)}$ such that $p_1(\vec{c}, \theta) < p_n(\vec{c}, \theta)$.

So, $E_{\theta > \frac{v(n-1)}{v(n)}}[p_n(\vec{c}, \theta)] > E_{\theta > \frac{v(n-1)}{v(n)}}[p_1(\vec{c}, \theta)]$ and $E_{\theta > \frac{v(n-1)}{v(n)}}[p_j(\vec{c}, \theta)] \geq E_{\theta > \frac{v(n-1)}{v(n)}}[p_1(\vec{c}, \theta)]$ for all $j \neq 1$. Since $E_{\theta > \frac{v(n-1)}{v(n)}}[p_1(\vec{c}, \theta)] + E_{\theta > \frac{v(n-1)}{v(n)}}[p_2(\vec{c}, \theta)] + ... + E_{\theta > \frac{v(n-1)}{v(n)}}[p_n(\vec{c}, \theta)] = 1$, we must have $E_{\theta > \frac{v(n-1)}{v(n)}}[p_1(\vec{c}, \theta)] < \frac{1}{n}$. The payoff to agent 1 for all $\theta \leq \frac{v(n-1)}{v(n)}$ is at most 1, therefore its payoff for deviating earlier, $E_\theta[p_1(\vec{c}, \theta)] \leq \Pr(\theta \leq \frac{v(n-1)}{v(n)}) + E_{\theta > \frac{v(n-1)}{v(n)}}[p_1(\vec{c}, \theta)]$. In order for this deviation not to be profitable for player 1, we need $E_\theta[p_1(\vec{c}, \theta)] < \frac{1}{n}$. Setting $\frac{v(n-1)}{v(n)} < \frac{1}{n} - E_{\theta > \frac{v(n-1)}{v(n)}}[p_1(\vec{c}, \theta)]$, gives us the desired result. Therefore, there exists a sufficiently complementary valuation such that the payoff of any player for deviating earlier is less than playing in round $t > 1$. Hence all players going in round t, for any $t > 1$, is supported by a subgame perfect Nash equilibrium.

□

5. Conclusions

We have introduced a game-theoretic model for the sequential revelation of answers by users who participate in question-and-answer forums. The best-answer rule, which models the rule used by Yahoo! Answers, is effective when the asker has a substitutes valuation function for information, but enables only the least efficient outcome, in which every user plays in the very last round, for complements valuations. In considering the effect of different rules on the equilibrium structure of the game, we have identified two rules (the approval-voting and proportional-share rules) that lead to efficient outcomes for subclasses of complements valuations. For substitutes valuations, the approval-voting rule can also introduce equilibria in which all users play last, while the proportional-share rule retains the efficient outcome in the unique equilibrium. Adopting an axiomatic approach, we establish that there does not exist a rule for assigning points to users based on answers that satisfies a set of reasonable axioms and isolates an equilibrium in which all users reveal their information immediately.

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Appendix

5.1. Approval-Voting Rule

Lemma 3.3. All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the approval-voting rule (with $k > 1$) for any complements valuations.

This result is established via strong induction on the total number of agents that play in the last two active rounds (in equilibrium).

Lemma 5.1. *No strategy profile (with at least two active rounds in equilibrium) that has $l \leq k$ players in the last two active rounds can be a subgame perfect Nash equilibrium with the approval-voting rule (with $k > 1$) for any valuation function.*

Proof. Consider the subgame corresponding to the penultimate active round. Suppose that i agents play in this round and j play in the last active round. The expected payoff (conditioned on reaching this subgame) of an agent in the last active round is $p_2 = \Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j+1}+...+\delta_n}{\delta_{n-j-i+1}+...+\delta_n}$. By deviating to the penultimate active round, its expected payoff would be 1 since the agent is then sure to be one of the last k agents to play before the question closes. Therefore, the player can profitably deviate. $\square$

From this, we can immediately establish the base case of our induction. The base case holds equally for both substitutes and complements valuations.

Lemma 5.2. *No strategy profile (with at least two active rounds in equilibrium) that has two players in the last two active rounds can be a subgame perfect Nash equilibrium under the approval-voting rule (with $k > 1$) for any valuation function.*

Proof. Immediate from Lemma 5.1 since $k > 1$. $\square$

Now we are ready for the inductive step. Define S_l as follows: *No strategy profile (with at least two active rounds in equilibrium) that has l players in the last two active rounds can be a subgame perfect Nash equilibrium with the approval-voting rule (with $k > 1$) for complements valuations.*

Lemma 5.3. *Given that statements $S_2, ..., S_l$ are true for $l < n$, then S_{l+1} is true.*

Proof. By Lemma 5.1 we can focus on the case of $k < l$. For ease of presentation, refer to the penultimate active round as round #1 and the last active round as round #2. Suppose i agents play in #1 and j in #2 (where $i + j = l + 1$.) For now, assume that $i \geq 2$. In all cases, consider the subgame (round #1) reached when all players are following the prescribed strategy, and condition on having reached this subgame.

Consider the case that $j = 1$. Since $k < i + j$, we can assume $i \geq k$. The expected payoff to an agent in #1 is $p_1 \cdot \frac{k}{i} + p_2 \cdot \frac{k-1}{i}$ where $p_1 = \Pr(\theta \leq \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j-i+1}+...+\delta_{n-j}}{\delta_{n-j-i+1}+...+\delta_n}$ and $p_2 = \Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j+1}+...+\delta_n}{\delta_{n-j-i+1}+...+\delta_n}$, and recognizing that the agent can compete for votes in the event that the question closes in #2. Consider a deviation to a later round by such an agent. By the inductive hypothesis, this would be a subgame with l or less players, and so all would play in the same round. The expected payoff would be $p_3 = \Pr(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j}+...+\delta_n}{\delta_{n-j-i+1}+...+\delta_n}$. For a SPNE we need $p_1 \cdot \frac{k}{i} + p_2 \cdot \frac{k-1}{i} \geq p_3$ (eq. 6). The expected payoff to the agent in #2 is p_2. Deviating to #1 would bring this agent payoff $\frac{k}{i+1}$ and so for a SPNE we need $p_2 \geq \frac{k}{i+1}$ (eq. 5), which becomes $\delta_n \geq (\delta_{n-i} + ... + \delta_n) \cdot \frac{k}{i+1}$. Eq. 6 becomes $(\delta_{n-i} + ... + \delta_{n-1}) \cdot \frac{k}{i} + \delta_n \cdot \frac{k-1}{i} \geq \delta_{n-1} + \delta_n$, and so $(\delta_{n-i} + ... + \delta_{n-1}) \cdot k + \delta_n \cdot k \geq i \cdot \delta_{n-1} + (i+1) \cdot \delta_n$, and so $(\delta_{n-i} + ... + \delta_n) \cdot k > (i+1) \cdot \delta_n$, and a contradiction with (eq. 5).

Continuing, we can now assume $j > 1$, along with $i \geq 2$.

Consider the case $i < k$ and $j \geq k$. The expected payoff to an agent in #1 is p_1 and deviating to a later round brings $p_3 \cdot \frac{k}{j+1}$ by a similar argument to above (and appeal to the inductive hypothesis.) For a SPNE, we need $v(n-j) - v(n-j-i) \geq (v(n) - v(n-j-1)) \frac{k}{j+1}$. But, $\frac{v(n-j)-v(n-j-i)}{k} < \frac{v(n-j)-v(n-j-i)}{i} < \frac{v(n)-v(n-j-1)}{j+1}$ where the first inequality follows since $i < k \leq j$ and the second by the complements property. Therefore a user in #1 has a profitable deviation.

Consider the case $i < k$ and $j < k$, still with $k < i + j$. Now, the expected payoff to an agent in #1 is $p_1 + p_2 \cdot \frac{k-j}{i}$, and to preclude a useful deviation we need $p_1 + p_2 \cdot \frac{k-j}{i} \geq p_3$ (eq. 1). By similar arguments to above, we also require $p_2 \geq p_4 + p_5 \cdot \frac{k-j+1}{i+1}$ (eq. 2), where $p_4 = \Pr(\theta \leq \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)})$ and $p_5 = \Pr(\theta > \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)})$ to preclude a useful deviation by an agent in #2 to #1. Eq. 1 becomes $(\delta_{n-j-i+1} + ... + \delta_{n-j}) + (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{k-j}{i} \geq \delta_{n-j} + ... + \delta_n$, or equivalently $\delta_{n-j+1} + ... + \delta_n \leq (\delta_{n-j-i+1} + ... + \delta_{n-j-1}) + (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{k-j}{i}$. Eq. 2 becomes $(\delta_{n-j-i+1} + ... + \delta_{n-j+1}) + (\delta_{n-j+2} + ... + \delta_n) \cdot \frac{k-j+1}{i+1} \leq \delta_{n-j+1} + ... + \delta_n$. Combining with Eq. 1 and canceling terms, this gives $\delta_{n-j} + \delta_{n-j+1} + (\delta_{n-j+2} + ... + \delta_n) \cdot \frac{k-j+1}{i+1} \leq (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{k-j}{i}$, but this is a contradiction because $k < i + j$ and so $\frac{k-j+1}{i+1} > \frac{k-j}{i}$.

Consider the case $i \geq k$ and $j \geq k$. The expected payoff to an agent in #1 is $p_1 \cdot \frac{k}{i}$ and to be a SPNE we require $p_1 \cdot \frac{k}{i} \geq p_3 \cdot \frac{k}{j+1}$. For this, we need $v(n-j) - v(n-j-i)\frac{k}{i} \geq (v(n) - v(n-j-1))\frac{k}{j+1}$, but $\frac{v(n-j)-v(n-j-i)}{i} < \frac{v(n)-v(n-j-1)}{j+1}$ for any complements valuation.

Consider the case $i \geq k$ and $j < k$. Considering a deviation by an agent in #1, we need $p_1 \cdot \frac{k}{i} + p_2 \frac{(k-j)}{i} \geq p_3$. This requires $(\delta_{n-j-i+1} + ... + \delta_{n-j}) \cdot \frac{k}{i} + (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{k-j}{i} \geq \delta_{n-j} + ... + \delta_n$, and so $(\delta_{n-j-i+1} + ... + \delta_{n-j}) \cdot \frac{k}{i} + (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{k-j}{i} > \delta_{n-j+1} + ... + \delta_n$, or equivalently, $(\delta_{n-j-i+1} + ... + \delta_{n-j}) \cdot \frac{k}{i} > (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{i+j-k}{i}$ (Eq. 3). Considering a deviation by an agent in #2, we also need $p_2 \geq p_4 \cdot \frac{k}{i+1} + p_5 \cdot \frac{k-j+1}{i+1}$. This requires $\delta_{n-j+1} + ... + \delta_n \geq (\delta_{n-j-i+1} + ... + \delta_{n-j+1}) \cdot \frac{k}{i+1} + (\delta_{n-j+2} + ... + \delta_n) \cdot \frac{k-j+1}{i+1}$, and so $\delta_{n-j+1} + ... + \delta_n > (\delta_{n-j-i+1} + ... + \delta_{n-j}) \cdot \frac{k}{i+1} + (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{k-j+1}{i+1}$ (since $j > 1$ and so $\frac{k}{i+1} < \frac{k-j+1}{i+1}$), or equivalently, $(\delta_{n-j+1} + ... + \delta_n) \cdot \frac{i+j-k}{i+1} > (\delta_{n-j-i+1} + ... + \delta_{n-j}) \cdot \frac{k}{i+1}$. We have a contradiction with Eq. 3, and so this cannot be part of a SPNE.

Finally, we must consider the case $i = 1$. From Lemma 5.4, we know this case cannot be a SPNE. □

Lemma 5.4. *No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and l players in the last active round can be a SPNE with the approval-voting rule (with $k > 1$) for any complements valuation function, given that no strategy profile is a SPNE when there are either (a) at most l players in the last two active rounds, or (b) exactly $l + 1$ players in the last two active rounds with at least two players in the penultimate active round.*

Proof. Let i be the player who participates in the penultimate active round. We will establish this via strong induction on the number of rounds before T where agent i plays. The expected payoff is conditioned throughout on reaching the penultimate active round.

Base Case: No strategy profile with at least two active rounds in equilibrium in which there is one player in the penultimate active round and this round is in period $T - 1$ can be a SPNE. Let j denote the number of players in the last active round. By Lemma 5.1 we can assume $j \geq k$. The expected payoff to player i is $\Pr(\theta \leq \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-1)}{v(n)}) = \frac{\delta_{n-j}}{\delta_{n-j} + ... + \delta_n}$, and deviating to play in round T brings expected payoff (conditioned on reaching the penultimate active round) of $\frac{k}{j+1}$. In order for this to be part of a subgame perfect Nash equilibrium, we need $\frac{\delta_{n-j}}{\delta_{n-j} + ... + \delta_n} \geq \frac{k}{j+1}$. But since $(\delta_{n-j} + ... + \delta_n) \cdot \frac{1}{j+1} \geq \delta_{n-j}$, for the case of complements, we know that $(\delta_{n-j} + ... + \delta_n) \cdot \frac{k}{j+1} > \delta_{n-j}$ (for $k > 1$), and thus this strategy profile cannot be a subgame perfect Nash equilibrium.

Inductive Hypothesis: No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round, and this round is $r+1$ periods before T can be a SPNE. Again, focus on the case that $j \geq k$ (because otherwise we can appeal to Lemma 5.1.) Consider what happens when the player in the penultimate active round deviates and goes later. By assumption (a) in the statement of the lemma we know that there can be at most two active rounds in the resulting subgame, because otherwise the last two active rounds would include l or less players. Then, by the inductive case for r periods to go and by assumption (b) in the statement of the lemma, the only SPNE in that subgame involves all players playing in the same round. Then, by the same analysis as for the base case, a player in the penultimate active round can profitably deviate to play in the same round with the other j players. This completes the proof. □

Lemma 3.6. All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the approval-voting rule (with $k > 1$) for any substitutes valuations.

This result is established via strong induction on the total number of agents that play in the last two active rounds (in equilibrium). Recall that an active round is a round in which at least one agent plays. The base case is already established above.

Now we are ready for the inductive step. Define S_l as follows: *No strategy profile (with at least two active rounds in equilibrium) that has l players in the last two active rounds can be a subgame perfect Nash equilibrium with the approval-voting rule (with $k > 1$) for substitutes valuations.*

Lemma 5.5. *Given that statements $S_2, ..., S_l$ are true, for $l < n$, then S_{l+1} is true.*

Proof. By Lemma 5.1 we can focus on the case $k < l$. For ease of presentation, refer to the penultimate active round as round #1 and the last active round as round #2. Suppose i agents play in #1 and j in #2 (where $i + j = l + 1$.) For now, assume that $i \geq 2$. In all cases, consider the subgame (round #1) reached when all players are following the prescribed strategy, and condition on having reached this subgame.

Consider the case that $j = 1$. Since $k < i + j$, we can assume $i \geq k$. The expected payoff to the agent in #2 is $p_2 = \Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j+1}+...+\delta_n}{\delta_{n-j-i+1}+...+\delta_n}$. Deviating to #1 would bring this agent payoff $\frac{k}{i+1}$. But, we have $p_2 = \frac{\delta_{n-j+1}+...+\delta_n}{\delta_{n-j-i+1}+...+\delta_n} < \frac{1}{i+1} < \frac{k}{i+1}$ and so a useful deviation, where the first inequality is by the substitutes property.

Now assume $j > 1$, along with $i \geq 2$.

Consider the case $i < k$ and $j > k$. The expected payoff to an agent in #1 is $p_1 = \Pr(\theta \leq \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j-i+1}+...+\delta_{n-j}}{\delta_{n-j-i+1}+...+\delta_n}$. Consider a deviation to a later round by such an agent. By the inductive hypothesis, this would be a subgame with l or less players, and so all would play in the same round. The expected payoff would be $\frac{k}{j+1}p_3$ with $p_3 = \Pr(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j}+...+\delta_n}{\delta_{n-j-i+1}+...+\delta_n}$. For a SPNE we need $p_1 \geq \frac{k}{j+1}p_3$ (eq. 6). The expected payoff to an agent in #2 is $\frac{k}{j}p_2$. Consider a deviation to round #1 by such an agent. By the inductive hypothesis, the remaining $j-1 \geq k$ agents would all play in the same round and so the expected payoff for a deviation would be $p_4 = \Pr(\theta \leq \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j-i+1}+...+\delta_{n-j+1}}{\delta_{n-j-i+1}+...+\delta_n}$ (since $i+1 \leq k$). For a SPNE we need $\frac{k}{j}p_2 \geq p_4$ (eq. 5). Eq. 5 becomes $\frac{\delta_{n-j+1}+...+\delta_n}{j} \geq \frac{\delta_{n-j-i+1}+...+\delta_{n-j+1}}{k}$, and so $\frac{\delta_{n-j+1}+...+\delta_n}{j} \geq \frac{\delta_{n-j-i+1}+...+\delta_{n-j}}{k}$. Eq. 6 becomes $\frac{\delta_{n-j-i+1}+...+\delta_{n-j}}{k} \geq \frac{\delta_{n-j}+...+\delta_n}{j+1}$. For any substitutes valuation, $\frac{\delta_{n-j}+...+\delta_n}{j+1} > \frac{\delta_{n-j+1}+...+\delta_n}{j}$, so both eq. 5 and eq. 6 cannot hold simultaneously.

Consider the case $i < k$ and $j = k$. To preclude a deviation by an agent in #1 we again need $p_1 \geq \frac{k}{j+1}p_3$ (eq. 4). But now when considering a deviation by an agent in #2 to #1, we must also observe that it can also benefit from a vote coming from the question closing when the remaining $k-1$ agents answer. Moreover, when playing in #2 the agent doesn't need to compete for a vote. For SPNE, we need $p_2 \geq p_4 + p_5 \cdot \frac{1}{i+1}$ (eq. 3), where $p_5 = \Pr(\theta > \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{\delta_{n-j+2}+...+\delta_n}{\delta_{n-j-i+1}+...+\delta_n}$. Eq. 3 becomes $(\delta_{n-j+1} + ... + \delta_n) \geq (\delta_{n-j-i+1} + ... + \delta_{n-j+1}) + (\delta_{n-j+2} + ... + \delta_n) \cdot \frac{1}{i+1}$, and so $(\delta_{n-j+1} + ... + \delta_n) \geq (\delta_{n-j-i+1}+...+\delta_{n-j})+(\delta_{n-j+1}+...+\delta_n) \cdot \frac{1}{i+1}$, or in other words, $(\delta_{n-j+1}+...+\delta_n) \cdot \frac{i}{i+1} \geq \delta_{n-j-i+1}+...+\delta_{n-j}$. Eq. 4 becomes $\delta_{n-j-i+1}+...+\delta_{n-j} \geq (\delta_{n-j}+...+\delta_n) \cdot \frac{k}{k+1}$, and so $\delta_{n-j-i+1}+...+\delta_{n-j} \geq (\delta_{n-j+1}+...+\delta_n) \cdot \frac{k}{k+1}$. Observing that $i < k$ and so $\frac{i}{i+1} < \frac{k}{k+1}$ we see that eq. 3 and eq. 4 cannot hold simultaneously.

Consider the case $i < k$ and $j < k$, still with $k < i + j$. Now, the expected payoff to an agent in #1 is $p_1 + p_2 \cdot \frac{k-j}{i}$ (recognizing that the agent can compete for votes in the event that the question closes in #2), and to preclude a useful deviation to a later round by an agent in #1, we require $p_1 + p_2 \cdot \frac{k-j}{i} \geq p_3$ (eq. 1). By similar arguments to above, we also require $p_2 \geq p_4 + p_5 \cdot \frac{k-j+1}{i+1}$ (eq. 2) to preclude a useful deviation by an agent in #2 to #1. Eq. 1 becomes $(\delta_{n-j-i+1} + ... + \delta_{n-j}) + (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{k-j}{i} \geq \delta_{n-j} + ... + \delta_n$, and so $\delta_{n-j-i+1} + ... + \delta_{n-j} > (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{i+j-k}{i}$. Eq. 2 becomes $\delta_{n-j+1} + ... + \delta_n \geq (\delta_{n-j-i+1} + ... + \delta_{n-j+1}) + (\delta_{n-j+2} + ... + \delta_n) \cdot \frac{k-j+1}{i+1}$, and so $\delta_{n-j+1} + ... + \delta_n > (\delta_{n-j-i+1} + ... + \delta_{n-j}) + (\delta_{n-j+1} + ... + \delta_n) \cdot \frac{k-j+1}{i+1}$ (since $k < i + j$), or in other words, $(\delta_{n-j+1} + ... + \delta_n) \cdot \frac{i+j-k}{i+1} > \delta_{n-j-i+1} + ... + \delta_{n-j}$. We have a contradiction, and eq. 1 and eq. 2 cannot hold simultaneously.

Consider the case $i \geq k$ and $j > 1$. The expected payoff to an agent in #2 is upper-bounded by $p_2 \cdot \frac{k}{j}$, since it is p_2 when $j \leq k$ and $p_2 \cdot \frac{k}{j}$ otherwise. By similar arguments to above, deviating to #1 will bring an agent at least $p_4 \cdot \frac{k}{i+1}$, since in the case where $j \leq k$ the expected payoff is $p_4 \cdot \frac{k}{i+1} + p_5 \cdot \frac{k-j+1}{i+1}$. So, in order to preclude a useful deviation we at least need $p_2 \cdot \frac{k}{j} \geq p_4 \cdot \frac{k}{i+1}$, or in other words, $(\delta_{n-j+1} + ... + \delta_n) \cdot \frac{k}{j} \geq (\delta_{n-j-i+1} + ... + \delta_{n-j+1}) \cdot \frac{k}{i+1}$. But $\frac{\delta_{n-j+1}+...+\delta_n}{j} < \frac{\delta_{n-j-i+1}+...+\delta_{n-j+1}}{i+1}$ for substitutes valuations, and so we have a contradiction.

Finally, when $i = 1$, Lemma 5.6 tells us that this cannot be a SPNE. □

Lemma 5.6. *No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and l players in the last active round can be a SPNE with the approval-voting rule (with $k > 1$) for any substitutes valuation function, given that no strategy profile is a SPNE when there are either (a) at most l players in the last two active rounds, or (b) exactly $l+1$ players in the last two active rounds with at least two players in the penultimate active round.*

Proof. Let i be the player who participates in the penultimate active round. We will establish this via strong induction on the number of rounds before T where agent i plays. The expected payoff is conditioned throughout on reaching the penultimate active round. Terms p_1, p_2, p_4 and p_5 are as defined in the proof of Lemma 3.6.

Base Case: No strategy profile with at least two active rounds in equilibrium in which there is one player in the penultimate active round and this round is in period $T-1$ can be a SPNE. Let j denote the number of players in the last active round. By Lemma 5.1 we can assume $j \geq k$. For $j > k$ we have expected payoff of p_1 and $p_2 \cdot \frac{k}{j}$ to an agent in #1 and #2 respectively. To preclude a deviation by an agent in #2 to #1 we need $p_2 \cdot \frac{k}{j} \geq p_4$. For an agent deviating from #1 to round T, we need $p_1 \geq \frac{k}{j+1}$. The first equation becomes: $(j+1-k) \cdot \delta_{n-j} \geq (\delta_{n-j+1} + ... + \delta_n) \cdot k$ and the second becomes $(\delta_{n-j+1} + ... + \delta_n) \cdot k \geq j \cdot (\delta_{n-j} + \delta_{n-j+1})$, which cannot hold simultaneously.

For $j = k$, we have expected payoff of p_1 and p_2 to an agent in #1 and #2 respectively. To prevent a deviation by an agent in #2 to #1 we need $p_2 \geq p_4 + p_5 \cdot \frac{1}{i+1}$. For an agent deviating from #1 to round T, her expected payoff becomes $\frac{k}{k+1}$, and to preclude a deviation we need $p_1 \geq \frac{k}{k+1}$ (eq. 7). Then, Eq. 8 is the same as $\delta_{n-j} \leq (\delta_{n-j+2} + ... + \delta_n) \cdot \frac{k-2}{k-1}$. Eq. 7 becomes $\delta_{n-j} \geq (\delta_{n-j} + ... + \delta_n) \cdot \frac{k}{k+1}$, and it is impossible for both eq. 7 and eq. 8 to hold simultaneously.

Inductive Hypothesis: No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and this round is $r+1$ periods before T can be a SPNE. Again, assume w.l.o.g. that $j \geq k$. Consider what happens when the player in the penultimate active round deviates and goes later. By assumption (a) in the statement of the lemma we know that there can be at most two active rounds in the resulting subgame, because otherwise the last two active rounds would include l or less players. Then, by the inductive case for r periods to go, and by assumption (b) in the statement of the lemma, the only SPNE in the subgame following the deviation to a later round involves all players playing in the same round. Then, by the same analysis as for the base case, either the player in the penultimate active round can profitably deviate later or a player in the last active round can profitably deviate to the penultimate active round. □

5.2. Proportional-Share Rule

Theorem 3.10 All agents still to play will play in the same round in the equilibrium play in every SPNE of every subgame under the proportional-share for any additive complements valuations.

This result is established via strong induction on the total number of agents that play in the last two active rounds (in equilibrium).

Lemma 5.7. *No strategy profile (with at least two active rounds in equilibrium) that has exactly two players in the last two active rounds can be a subgame perfect Nash equilibrium under the proportional-share rule for any additive complements valuation function.*

Proof. Consider the subgame corresponding to the penultimate active round. The expected payoff (conditioned on reaching this subgame) of an agent in the last active round is $\Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) \cdot \frac{v(n)-v(n-1)}{v(n)} = \frac{v(n)-v(n-1)}{v(n)-v(n-2)} \frac{v(n)-v(n-1)}{v(n)}$. By deviating to the penultimate active round, this player's expected payoff becomes $\frac{v(n)-v(n-2)}{v(n)-v(n-2)}\frac{v(n)-v(n-2)}{2v(n)}$. In order for this to be an equilibrium, we need: $\frac{(v(n)-v(n-1))^2}{v(n)(v(n)-v(n-2))} \geq \frac{(v(n)-v(n-2))^2}{2v(n)(v(n)-v(n-2))}$, which is equivalent for additive complements to: $4(2n)^2 \geq (2n+2(n-1))^2$ or $0 \geq 2n^2-4n+1$. The right hand side is greater than 0 for all $n > 2$ since the roots of $2n^2-4n+1$ are $\approx 0.3, 1.7$. Therefore, the player in the last active round can profitably deviate. □

Now we are ready for the inductive step. Define S_l as follows: *No strategy profile (with at least two active rounds in equilibrium) that has exactly l players in the last two active rounds can be a subgame perfect Nash equilibrium under the proportional-share rule for any additive complements valuations.*

Lemma 5.8. *Given that statements $S_2, ..., S_l$ are true for $l < n$, then S_{l+1} is true.*

Proof. For ease of presentation, refer to the penultimate active round as round #1 and the last active round as round #2. Suppose i agents play in #1 and j in #2 (where $i+j=l+1$.) For now, assume that $i \geq 2$. In all cases, consider the subgame (round #1) reached when all players are following the prescribed strategy, and condition on having reached this subgame.

First assume that $i > 1$. The expected payoff to an agent in round #1 is $p_1 \cdot \frac{v(n-j)-v(n-j-i)}{i \cdot v(n-j)} + p_2 \cdot \frac{v(n-j)-v(n-j-i)}{i \cdot v(n)}$, where $p_1 = \Pr(\theta \leq \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n-j)-v(n-j-i)}{v(n)-v(n-j-i)}$ and $p_2 = \Pr(\theta > \frac{v(n-j)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n)-v(n-j)}{v(n)-v(n-j-i)}$. Consider a deviation to a later round by such an agent. By the inductive hypothesis, this would be a subgame with l or less players, and so all would play in the same round. Therefore, the expected payoff for such a deviation would be $p_3 \cdot \frac{v(n)-v(n-j-1)}{(j+1)v(n)}$ and $p_3 = \Pr(\theta > \frac{v(n-j-1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n)-v(n-j-1)}{v(n)-v(n-j-i)}$. The expected payoff to an agent in round #2 is $p_2 \cdot \frac{v(n)-v(n-j)}{j \cdot v(n)}$. Consider a deviation to round #1 by such an agent. By the inductive hypothesis, the remaining $j-1 \leq l$ players would all play in the same round and so the expected payoff for a deviation would be $p_4 \cdot \frac{v(n-j+1)-v(n-j-i)}{(i+1)v(n-j+1)} + p_5 \cdot \frac{v(n-j+1)-v(n-j-i)}{(i+1)v(n)}$, where $p_4 = \Pr(\theta \leq \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n-j+1)-v(n-j-i)}{v(n)-v(n-j-i)}$ and $p_5 = \Pr(\theta > \frac{v(n-j+1)}{v(n)} | \theta > \frac{v(n-j-i)}{v(n)}) = \frac{v(n)-v(n-j+1)}{v(n)-v(n-j-i)}$. For an SPNE we require $\frac{v(n-j)-v(n-j-i)}{v(n)-v(n-j-i)} \cdot \frac{v(n-j)-v(n-j-i)}{i \cdot v(n-j)} + \frac{v(n)-v(n-j)}{v(n)-v(n-j-i)} \cdot \frac{v(n-j)-v(n-j-i)}{i \cdot v(n)} \geq \frac{(v(n)-v(n-j-1))^2}{(j+1)v(n)(v(n)-v(n-j-i))}$ and $\frac{(v(n)-v(n-j))^2}{j \cdot v(n)(v(n)-v(n-j-i))} \geq \frac{v(n-j+1)-v(n-j-i)}{v(n)-v(n-j-i)} \cdot \frac{v(n-j+1)-v(n-j-i)}{(i+1)v(n-j+1)} + \frac{v(n)-v(n-j+1)}{v(n)-v(n-j-i)} \cdot \frac{v(n-j+1)-v(n-j-i)}{(i+1)v(n)}$. Therefore, for an SPNE, we know that $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j-i)}{v(n-j)} - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j-i)}{v(n-j+1)} - \frac{v(n-j+1)}{v(n)}) \geq \frac{(v(n)-v(n-j-1))^2}{(j+1)(v(n))} - \frac{(v(n)-v(n-j))^2}{j(v(n))}$. In order to establish a contradiction, we show that this equation can never hold.

First, we observe that for additive complements, we have $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j-i)}{v(n-j)} - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j-i)}{v(n-j+1)} - \frac{v(n-j+1)}{v(n)}) < \frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j+1)}{v(n)})$. For this, we need $\frac{v(n-j+1)-v(n-j-i)}{(i+1)v(n-j+1)} < \frac{v(n-j)-v(n-j-i)}{i \cdot v(n-j)}$. By additive complements, we require $\frac{2(n-j+1)-i}{(n-j+1)(n-j+2)} < \frac{2(n-j)-i+1}{(n-j)(n-j+1)}$, which is equivalent to $\frac{i+2}{n-j+2} > \frac{i-1}{n-j}$. Now, we have $\frac{i+2}{n-j+2} \geq \frac{i}{n-j} > \frac{i-1}{n-j}$, since $i \leq n-j$. Based on this, it is sufficient for a contradiction to establish $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j+1)}{v(n)}) < \frac{(v(n)-v(n-j-1))^2}{(j+1)(v(n))} - \frac{(v(n)-v(n-j))^2}{j(v(n))}$. This is equivalent to: $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2v(n)-v(n-j)) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2v(n)-v(n-j+1)) < \frac{(v(n)-v(n-j-1))^2}{(j+1)} - \frac{(v(n)-v(n-j))^2}{j}$.

Note that the following identities hold for additive complements:

$$\frac{v(n-j)-v(n-j-i)}{i} \cdot (2v(n)-v(n-j)) = (2n-2j-i+1)(2n(n+1)-(n-j)(n-j+1))$$

$$\frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2v(n)-v(n-j+1)) = (2n-2j-i+2)(2n(n+1)-(n-j+1)(n-j+2))$$

$$\frac{(v(n)-v(n-j))^2}{j} = j(2n-j+1)^2$$

$$\frac{(v(n)-v(n-j-1))^2}{j+1} = (j+1)(2n-j)^2$$

Therefore we write: $\frac{(v(n)-v(n-j-1))^2}{(j+1)} - \frac{(v(n)-v(n-j))^2}{j} = (j+1)(2n-j)^2 - j(2n-j+1)^2 = 4(n-j)^2 - j(j+1)$. And we can write: $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2v(n) - v(n-j)) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2v(n) - v(n-j+1)) = (2n-2j-i+1)(2n(n+1) - (n-j)(n-j+1)) - (2n-2j-i+2)(2n(n+1) - (n-j+1)(n-j+2)) = 2(2n-2j-i+1)(n-j+1) - 2n(n+1) + (n-j+1)(n-j+2)$. Left to prove is

$$2(2n-2j-i+1)(n-j+1) - 2n(n+1) + (n-j+1)(n-j+2) < 4(n-j)^2 - j(j+1) \tag{1}$$

The LHS is maximized for $i = 1$, and substituting for this it suffices to show $4(n-j)^2 + 4(n-j) + (n-j+1)(n-j+2) - 2n(n+1) < 4(n-j)^2 - j(j+1)$, or in other words: $j(j+1) + 4(n-j) + (n-j+1)(n-j+2) - 2n(n+1) < 0$, which is equivalent to $2(j-1)^2 - 2j(n+1) < n^2 - 5n$. Note that the LHS is negative for all values of i and n since $j \geq 1$, and the RHS is non-negative for $n \geq 5$. Therefore we know that this equation holds for all $n \geq 5$ and all $i + j \leq n$. When $n = 4$ this equation becomes, $2(j-1)^2 - 10j < -4$ or $2j^2 - 14j + 6 < 0$. $2j^2 - 14j + 6$ is less than 0 for all $1 \leq j \leq 3$. When $n = 3$, this equation becomes $2j^2 - 12j + 8 < 0$. $2j^2 - 12j + 8$ is less than 0 for all $1 \leq j \leq 2$. Therefore we have established the desired result.

Finally, when $i = 1$, Lemma 5.9 tells us that this cannot be a SPNE. □

Lemma 5.9. *No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and l players in the last active round can be a SPNE with the proportional-share rule for any additive complements valuation function, given that no strategy profile is a SPNE when there are either (a) at most l players in the last two active rounds, or (b) exactly $l+1$ players in the last two active rounds with at least two players in the penultimate active round.*

Proof. Let i be the player who participates in the penultimate active round. We will establish this via strong induction on the number of rounds before T where agent i plays. The expected payoff is conditioned throughout on reaching the penultimate active round.

Base Case: No strategy profile with at least two active rounds in equilibrium in which there is one player in the penultimate active round and this round is in period $T-1$ can be a SPNE. Let j denote the number of players in the last active round. Because we are focused on rounds $T-1$ and T then the same analysis as was used for the $i > 1$ case in the proof of Lemma 5.8 is valid here. Either the agent in the penultimate active round can usefully deviate later (in which case it necessarily plays in the same round as the other players since it plays in round T), or an agent in round T can usefully deviate and play in round $T-1$ with the singleton agent. For this, it is sufficient to note that the proof of Lemma 5.8 establishes that: $\frac{v(n-j)-v(n-j-i)}{i} \cdot (2 - \frac{v(n-j-i)}{v(n-j)} - \frac{v(n-j)}{v(n)}) - \frac{v(n-j+1)-v(n-j-i)}{i+1} \cdot (2 - \frac{v(n-j-i)}{v(n-j+1)} - \frac{v(n-j+1)}{v(n)}) < \frac{(v(n)-v(n-j-1))^2}{(j+1)(v(n))} - \frac{(v(n)-v(n-j))^2}{j(v(n))}$ for all i, j, n such that $i, j \geq 1$ and $i + j \leq n$.

Inductive Hypothesis: No strategy profile (with at least two active rounds in equilibrium) in which there is one player in the penultimate active round and this round is $r+1$ periods before T can be a SPNE. Consider what happens when the player in the penultimate active round deviates and goes later. By assumption (a) in the statement of the lemma we know that there can be at most two active rounds in the resulting subgame, because otherwise the last two active rounds would include l or less players. Then, by the inductive case for r periods to go, and by assumption (b) in the statement of the lemma, the only SPNE in the subgame following the deviation to a later round involves all players playing in the same round. Then, by the same analysis as for the base case, either the player in the penultimate active round can profitably deviate later or a player in the last active round can profitably deviate to the penultimate active round. □