Tree Automata Techniques and Applications

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Introduction

During the past few years, several of us have been asked many times about references on finite tree automata. On one hand, this is the witness of the liveness of this field. On the other hand, it was difficult to answer. Besides several excellent survey chapters on more specific topics, there is only one monograph devoted to tree automata by Gécseg and Steinby. Unfortunately, it is now impossible to find a copy of it and a lot of work has been done on tree automata since the publication of this book. Actually using tree automata has proved to be a powerful approach to simplify and extend previously known results, and also to find new results. For instance recent works use tree automata for application in abstract interpretation using set constraints, rewriting, automated theorem proving and program verification.

Tree automata have been designed a long time ago in the context of circuit verification. Many famous researchers contributed to this school which was headed by A. Church in the late 50's and the early 60's: B. Trakhtenbrot, J.R. Büchi, M.O. Rabin, Doner, Thatcher, etc. Many new ideas came out of this program. For instance the connections between automata and logic. Tree automata also appeared first in this framework, following the work of Doner, Thatcher and Wright. In the 70's many new results were established concerning tree automata, which lose a bit their connections with the applications and were studied for their own. In particular, a problem was the very high complexity of decision procedures for the monadic second order logic. Applications of tree automata to program verification revived in the 80's, after the relative failure of automated deduction in this field. It is possible to verify temporal logic formulas (which are particular Monadic Second Order Formulas) on simpler (small) programs. Automata, and in particular tree automata, also appeared as an approximation of programs on which fully automated tools can be used. New results were obtained connecting properties of programs or type systems or rewrite systems with automata.

Our goal is to fill in the existing gap and to provide a textbook which presents the basics of tree automata and several variants of tree automata which have been devised for applications in the aforementioned domains. We shall discuss only *finite tree* automata, and the reader interested in infinite trees should consult any recent survey on automata on infinite objects and their applications (See the bibliography). The second main restriction that we have is to focus on the operational aspects of tree automata. This book should appeal the reader who wants to have a simple presentation of the basics of tree automata, and to see how some variations on the idea of tree automata have provided a nice tool for solving difficult problems. Therefore, specialists of the domain probably know almost all the material embedded. However, we think that this book can be helpful for many researchers who need some knowledge on tree automata. This is typically the case of PhD a student who may find new ideas and guess connections with his (her) own work.

Again, we recall that there is no presentation nor discussion of tree automata for infinite trees. This domain is also in full development mainly due to applications in program verification and several surveys on this topic do exist. We have tried to present a tool and the algorithms devised for this tool. Therefore, most of the proofs that we give are constructive and we have tried to give as many complexity results as possible. We don't claim to present an exhaustive description of all possible finite tree automata already presented in the literature and we did some choices in the existing menagerie of tree automata. Although some works are not described thoroughly (but they are usually described in exercises), we think that the content of this book gives a good flavor of what can be done with the simple ideas supporting tree automata.

This book is an open work and we want it to be as interactive as possible. Readers and specialists are invited to provide suggestions and improvements. Submissions of contributions to new chapters and improvements of existing ones are welcome.

Among some of our choices, let us mention that we have not defined any precise language for describing algorithms which are given in some pseudo algorithmic language. Also, there is no citation in the text, but each chapter ends with a section devoted to bibliographical notes where credits are made to the relevant authors. Exercises are also presented at the end of each chapter.

Tree Automata and Their Applications is composed of six main chapters (numbered 1– 6). The first one presents tree automata and defines recognizable tree languages. The reader will find the classical algorithms and the classical closure properties of the class of recognizable tree languages. Complexity results are given when they are available. The second chapter gives alternative presentation of recognizable tree languages which may be more relevant in some situations. This includes regular tree grammars, regular tree expressions and regular equations. The description of properties relating regular tree languages and context-free word languages form the last part of this chapter. In Chapter 3, we show the deep connections between logic and automata. In particular, we prove in full details the correspondence between finite tree automata and the weak monadic second order logic with k successors. We also sketch several applications in various domains.

Chapter 4 presents a basic variation of automata, more precisely automata with equality constraints. An equality constraint restricts the application of rules to trees where some subtrees are equal (with respect to some equality relation). Therefore we can discriminate more easily between trees that we want to accept and trees that we must reject. Several kinds of constraints are described, both originating from the problem of non-linearity in trees (the same variable may occur at different positions).

In Chapter 5 we consider automata which recognize sets of sets of terms. Such automata appeared in the context of set constraints which themselves are used in program analysis. The idea is to consider, for each variable or each predicate symbol occurring in a program, the set of its possible values. The program gives constraints that these sets must satisfy. Solving the constraints gives an upper approximation of the values that a given variable can take. Such an approximation can be used to detect errors at compile time: it acts exactly as a typing system which would be inferred from the program. Tree set automata (as we call them) recognize the sets of solutions of such constraints (hence sets of sets of trees). In this chapter we study the properties of tree set automata and their relationship with program analysis.

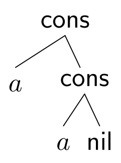
Originally, automata were invented as an intermediate between function description and their implementation by a circuit. The main related problem in the sixties was the *synthesis problem*: which arithmetic recursive functions can be achieved by a circuit ? So far, we only considered tree automata which accepts sets of trees or sets of tuples of trees (Chapter 3 or sets of sets of trees (Chapter 5). However, tree automata can also be used as a computational device. This is the subject of Chapter 6 where we study *tree transducers*.

Preliminaries

Signature, Terms and Contexts

A ***ranked alphabet*** is a couple $(\mathcal{F}, Arity)$ where $\mathcal{F}$ is a finite set and $Arity$ is a mapping from $\mathcal{F}$ into $\mathbb{N}$. The ***arity*** of a symbol $f \in \mathcal{F}$ is $Arity(f)$. The set of symbols of arity p is denoted by $\mathcal{F}_p$. Elements of arity 0, 1, … p are respectively called constants, unary, … p-ary symbols. We assume that $\mathcal{F}$ contains at least one constant. In the examples, we use parenthesis and commas for a short declaration of symbols with arity. For instance, $f(,)$ is a short declaration for a binary symbol f.

Example 1. Let $\mathcal{F} = \{\mathsf{cons}(,), \mathsf{nil}, a\}$. Here cons is a binary symbol, nil and a are constants. A term $\mathsf{cons}(a, \mathsf{cons}(a, \mathsf{nil}))$ is also represented in a graphical way:



Let $\mathcal{X}$ be a set of symbols called ***variables***. The set $T(\mathcal{F}, \mathcal{X})$ of ***terms*** over the ranked alphabet $\mathcal{F}$ and the set of variables $\mathcal{X}$ is the smallest set defined by:
- $\mathcal{F}_0 \subseteq T(\mathcal{F}, \mathcal{X})$ and
- $X \subseteq T(\mathcal{F}, \mathcal{X})$ and
- if $p \geq 1$, $f \in \mathcal{F}_p$ and $t_1, \ldots, t_p \in T(\mathcal{F}, \mathcal{X})$, then $f(t_1, \ldots, t_p) \in T(\mathcal{F}, \mathcal{X})$.

If $\mathcal{X} = \emptyset$ then $T(\mathcal{F}, \mathcal{X})$ is also written $T(\mathcal{F})$. Terms in $T(\mathcal{F})$ are called ***ground terms***. A term in $T(\mathcal{F}, \mathcal{X})$ is ***linear*** if each variable occurs at most once in t.

Let $\mathcal{X}_n$ be a set of n variables. A linear term $C \in T(\mathcal{F}, \mathcal{X}_n)$ is called a ***context*** and the expression $C[t_1, \ldots, t_n]$ for $t_1, \ldots, t_n \in T(\mathcal{F})$ denotes the term in $T(\mathcal{F})$ obtained from C by replacing for each $1 \leq i \leq n$ variable x_i by t_i. We denote by $\mathcal{C}^n(\mathcal{F})$ the set of contexts over $(x_1, \ldots, x_n)$ and $\mathcal{C}(\mathcal{F})$ the set of contexts containing a single variable.

$\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}^*$ denotes the set of finite strings over $\mathbb{N}$.

A finite ordered ***tree*** t over a set of labels E is a mapping from a prefix-closed set $\mathcal{P}os(t) \subseteq \mathbb{N}^*$ into E. Thus, a term $t \in T(\mathcal{F}, \mathcal{X})$ may be viewed as a finite ordered tree, the leaves of which are labeled with variables or constant symbols and the internal nodes are labeled with symbols of positive arity, with out-degree equal to the arity of the label, i.e. a term $t \in T(\mathcal{F}, \mathcal{X})$ can also be defined as a partial function $t : \mathbb{N}^* \to \mathcal{F} \cup \mathcal{X}$ with domain $\mathcal{P}os(t)$ satisfying the following properties:

- $\mathcal{P}os(t)$ is nonempty and prefix-closed.
- For each $p \in \mathcal{P}os(t)$, if $t(p) \in \mathcal{F}_n$, then $\{i|pi \in \mathcal{P}os(t)\} = \{1, \ldots, n\}$.
- For each $p \in \mathcal{P}os(t)$, if $t(p) \in \mathcal{X}$, then $\{i|pi \in \mathcal{P}os(t)\} = \emptyset$.

Each element of $\mathcal{P}os(t)$ is called a ***position***. A ***frontier position*** is a position p such that $\forall \alpha \in \mathbb{N}$, $p\alpha \notin \mathcal{P}os(t)$. The set of frontier positions is denoted by $\mathcal{FP}os(t)$. Each position p in t such that $t(p) \in \mathcal{X}$ is called a ***variable position***. The set of variable positions of p is denoted by $\mathcal{VP}os(t)$.

A ***subterm*** $t|_p$ of a term $t \in T(\mathcal{F}, \mathcal{X})$ at position p is defined by the following:

- $\mathcal{P}os(t|_p) = \{i \mid pi \in \mathcal{P}os(t)\}$,
- $\forall j \in \mathcal{P}os(t|_p)$, $t|_p(j) = t(pj)$.

We denote by $t[u]_p$ the term obtained by replacing in t the subterm $t|_p$ by u. We have $\mathcal{H}ead(t) = f$ if and only if $t(\varepsilon) = f$, that is f is the ***root symbol*** of t.

Functions on terms

The ***size*** of a term t, denoted by $\|t\|$ and the ***height*** of t, denoted by $\mathcal{H}eight(t)$ are inductively defined by:

- $\mathcal{H}eight(t) = 0$, $\|t\| = 0$ if $t \in \mathcal{X}$,
- $\mathcal{H}eight(t) = 1$, $\|t\| = 1$ if $t \in \mathcal{F}_0$,
- $\mathcal{H}eight(t) = 1+\max(\{\mathcal{H}eight(t_i) \mid i \in \{1, \ldots, n\}\})$, $\|t\| = 1+\sum_{i \in \{1,\ldots,n\}} \|t_i\|$ if $\mathcal{H}ead(t) \in \mathcal{F}_n$.

We denote by $\trianglerighteq$ the ***subterm ordering***, i.e. we write $t \trianglerighteq t'$ if t' is a subterm of t. We denote $t \triangleright t'$ if $t \trianglerighteq t'$ and $t \neq t'$. A set of terms F is said to be ***closed*** if it is closed under the subterm ordering, i.e. $\forall t \in F$ $(t \trianglerighteq t' \Rightarrow t' \in F)$.

A ***substitution*** (respectively a ***ground substitution***) σ is a mapping from $\mathcal{X}$ into $T(\mathcal{F}, \mathcal{X})$ (respectively into $T(\mathcal{F})$) where there are only finitely many variables not mapped to themselves. The ***domain*** of a substitution σ is the subset of variables $x \in \mathcal{X}$ such that $\sigma(x) \neq x$. The substitution $\{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$ is the identity on $\mathcal{X} \setminus \{x_1, \ldots, x_n\}$ and maps $x_i \in \mathcal{X}$ on $t_i T(\mathcal{F}, \mathcal{X})$, for every index $1 \leq i \leq n$. Substitutions can be extended to $T(\mathcal{F}, \mathcal{X})$ in such a way that:

$$\forall f \in \mathcal{F}_n, \forall t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{X}) \quad \sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n)).$$

We confuse a substitution and its extension to $T(\mathcal{F}, \mathcal{X})$. Substitutions will often be used in postfix notation: $t\sigma$ is the result of applying σ to the term t.

Chapter 1

Recognizable Tree Languages and Finite Tree Automata

In this chapter, we present basic results on finite tree automata in the style of the undergraduate textbook on finite automata by Hopcroft and Ullman [HU79]. We assume that the reader is familiar with finite automata. Words over finite alphabet can be viewed as unary terms. For instance a word abb over $A = \{a, b\}$ can be viewed as a unary term $t = a(b(b(\sharp)))$ over the ranked alphabet $\mathcal{F} = \{a(), b(), \sharp\}$ where $\sharp$ is a new constant symbol. The theory of tree automata arises as a straightforward extension of the theory of word automata when words are viewed as unary terms.

In Section 1.1, we define bottom-up finite tree automata where "bottom-up" has the following sense: assuming a graphical representation of trees or ground terms with the root symbol at the top, an automaton starts its computation at the leaves and moves upward. Recognizable tree languages are the languages recognized by some finite tree automata. We consider the deterministic case and the nondeterministic case and prove the equivalence. In Section 1.2, we prove a pumping lemma for recognizable tree languages. This lemma is useful for proving that some tree languages are not recognizable. In Section 1.3, we prove the basic closure properties for set operations. In Section 1.4, we define tree homomorphisms and study the closure properties under these tree transformations. In this Section the first difference between the word case and the tree case appears. Indeed, if recognizable word languages are closed under homomorphisms, recognizable tree languages are closed only under a subclass of tree homomorphisms: linear homomorphisms, where duplication of trees is forbidden. We will see all along this textbook that non linearity is one of the main difficulty for the tree case. In Section 1.5, we prove a Myhill-Nerode Theorem for tree languages and the existence of a unique minimal automaton. In Section 1.6, we define top-down tree automata. A second difference appears with the word case because it is proved that deterministic top-down tree automata are strictly less powerful than nondeterministic ones. The last section of the present chapter gives a list of complexity results.

1.1 Finite Tree Automata

Nondeterministic Finite Tree Automata

A ***finite Tree Automaton*** (NFTA) over $\mathcal{F}$ is a tuple $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ where Q is a set of (unary) states, $Q_f \subseteq Q$ is a set of final states, and Δ is a set of transition rules of the following type :

$$f(q_1(x_1), \dots, q_n(x_n)) \to q(f(x_1, \dots, x_n)),$$

where $n \geq 0$, $f \in \mathcal{F}_n$, $q, q_1, \dots, q_n \in Q$, $x_1, \dots, x_n \in \mathcal{X}$.

Tree automata over $\mathcal{F}$ run on ground terms over $\mathcal{F}$. An automaton starts at the leaves and moves upward, associating along a run a state with each subterm inductively. Let us note that there is no initial state in a NFTA, but, when $n = 0$, i.e. when the symbol is a constant symbol a, a transition rule is of the form $a \to q(a)$. Therefore, the transition rules for the constant symbols can be considered as the "initial" rules. If the direct subterms $u_1, \dots, u_n$ of $t = f(u_1, \dots, u_n)$ are labeled with states $q_1, \dots, q_n$, then the term t will be labeled by some state q with $f(q_1(x_1), \dots, q_n(x_n)) \to q(f(x_1, \dots, x_n)) \in \Delta$. We now formally define the move relation defined by an NFTA.

Let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ be an NFTA over $\mathcal{F}$. The ***move relation*** $\to_{\mathcal{A}}$ is defined by: let $t, t' \in T(\mathcal{F} \cup Q)$,

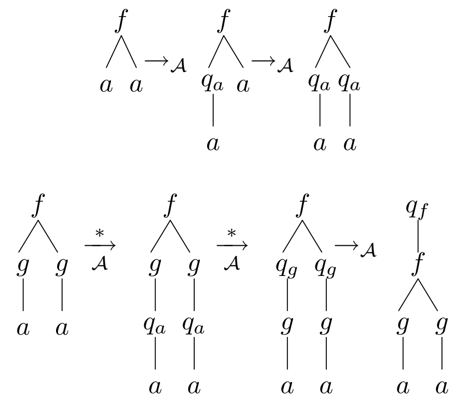
$$t \underset{\mathcal{A}}{\to} t' \Leftrightarrow \begin{cases} \exists C \in \mathcal{C}(\mathcal{F} \cup Q), \exists u_1, \dots, u_n \in T(\mathcal{F}), \\ \exists f(q_1(x_1), \dots, q_n(x_n)) \to q(f(x_1, \dots, x_n)) \in \Delta, \\ t = C[f(q_1(u_1), \dots, q_n(u_n))], \\ t' = C[q(f(u_1, \dots, u_n))]. \end{cases}$$

$\xrightarrow[\mathcal{A}]{*}$ is the reflexive and transitive closure of $\to_{\mathcal{A}}$.

Example 2. Let $\mathcal{F} = \{f(,), g(), a\}$. Consider the automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = \{q_a, q_g, q_f\}$, $Q_f = \{q_f\}$, and Δ is the following set of transition rules:

$$\begin{array}{lrcllcl} \{ & a & \to & q_a(a) & g(q_a(x)) & \to & q_g(g(x)) \\ & g(q_g(x)) & \to & q_g(g(x)) & f(q_g(x), q_g(y)) & \to & q_f(f(x,y)) \quad \} \end{array}$$

We give two examples of reductions with the move relation $\to_{\mathcal{A}}$



A ground term t in $T(\mathcal{F})$ is ***accepted*** by a finite tree automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ if

$$t \xrightarrow[\mathcal{A}]{*} q(t)$$

for some state q in Q_f. The reader should note that our definition corresponds to the notion of nondeterministic finite tree automaton because our finite tree automaton model allows zero, one or more transition rules with the same left-hand side. Therefore there are possibly more than one reduction starting with the same ground term. And, a ground term t is accepted if there is one reduction (among all possible reductions) starting from this ground term and leading to a configuration of the form $q(t)$ where q is a final state. The tree language $L(\mathcal{A})$ ***recognized*** by $\mathcal{A}$ is the set of all ground terms accepted by $\mathcal{A}$. A set L of ground terms is ***recognizable*** if $L = L(\mathcal{A})$ for some NFTA $\mathcal{A}$. The reader should also note that when we talk about the set recognized by a finite tree automaton $\mathcal{A}$ we are referring to the specific set $L(\mathcal{A})$, not just any set of ground terms all of which happen to be accepted by $\mathcal{A}$. Two NFTA are said to be ***equivalent*** if they recognize the same tree languages.

Example 3. Let $\mathcal{F} = \{f(,), g(), a\}$. Consider the automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = \{q, q_g, q_f\}$, $Q_f = \{q_f\}$, and $\Delta =$

$$\begin{array}{lrcl rcl}
\{ & a & \rightarrow & q(a) & g(q(x)) & \rightarrow & q(g(x)) \\
 & g(q(x)) & \rightarrow & q_g(g(x)) & g(q_g(x)) & \rightarrow & q_f(g(x)) \\
 & f(q(x), q(y)) & \rightarrow & q(f(x, y)) & \}. & &
\end{array}$$

We now consider a ground term t and exhibit three different reductions of term t w.r.t. move relation $\rightarrow_{\mathcal{A}}$.

$$\begin{array}{lcl cl}
t = g(g(f(g(a), a))) & \xrightarrow[\mathcal{A}]{*} & g(g(f(q_g(g(a)), q(a)))) & & \\
t = g(g(f(g(a), a))) & \xrightarrow[\mathcal{A}]{*} & g(g(q(f(g(a), a)))) & \xrightarrow[\mathcal{A}]{*} & q(t) \\
t = g(g(f(g(a), a))) & \xrightarrow[\mathcal{A}]{*} & g(g(q(f(g(a), a)))) & \xrightarrow[\mathcal{A}]{*} & q_f(t)
\end{array}$$

The term t is accepted by $\mathcal{A}$ because of the third reduction. It is easy to prove that $L(\mathcal{A})$ is the set of ground instances of $g(g(x))$.

The set of transition rules of a NFTA $\mathcal{A}$ can also be defined as a ground rewrite system, i.e. a set of ground transition rules of the form: $f(q_1, \ldots, q_n) \rightarrow q$. A move relation $\rightarrow_{\mathcal{A}}$ can be defined like previously. The only difference is that, now, we "forget" the ground subterms. And, a term t is accepted by a NFTA $\mathcal{A}$ if

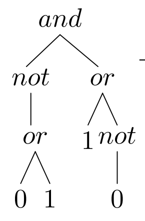
$$t \xrightarrow[\mathcal{A}]{*} q$$

for some final state q in Q_f. Unless it is stated otherwise, we will now refer to the definition with a set of ground transition rules. Considering a reduction starting from a ground term t and leading to a state q with the move relation, it is useful to remember the "history" of the reduction, i.e. to remember in which states are reduced the ground subterms of t. For this, we will adopt the following definitions. Let t be a ground term and $\mathcal{A}$ be a NFTA, a ***run*** r of $\mathcal{A}$ on t is a mapping $r : \mathcal{P}os(t) \to Q$ compatible with Δ, i.e. for every position p in $\mathcal{P}os(t)$, if $t(p) = f \in \mathcal{F}_n$, $r(p) = q$, $r(pi) = q_i$ for each $i \in \{1, \dots, n\}$, then $f(q_1, \dots, q_n) \to q \in \Delta$. A run r of $\mathcal{A}$ on t is ***successful*** if $r(\epsilon)$ is a final state. And a ground term t is accepted by a NFTA $\mathcal{A}$ if there is a successful run r of $\mathcal{A}$ on t.

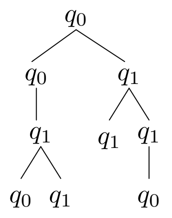
Example 4. Let $\mathcal{F} = \{or(,), and(,), not(), 0, 1\}$. Consider the automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = \{q_0, q_1\}$, $Q_f = \{q_1\}$, and $\Delta =$

$$\begin{array}{lrclrcll}
\{ & 0 & \to & q_0 & 1 & \to & q_1 & \\
 & not(q_0) & \to & q_1 & not(q_1) & \to & q_0 & \\
 & and(q_0,q_0) & \to & q_0 & and(q_0,q_1) & \to & q_0 & \\
 & and(q_1,q_0) & \to & q_0 & and(q_1,q_1) & \to & q_1 & \\
 & or(q_0,q_0) & \to & q_0 & or(q_0,q_1) & \to & q_1 & \\
 & or(q_1,q_0) & \to & q_1 & or(q_1,q_1) & \to & q_1 & \}.
\end{array}$$

A ground term over $\mathcal{F}$ can be viewed as a boolean formula without variable and a run on such a ground term can be viewed as the evaluation of the corresponding boolean formula. For instance, we give a reduction for a ground term t and the corresponding run given as a tree



$\xrightarrow[\mathcal{A}]{*} q_0$. ; the run r:



The tree language recognized by $\mathcal{A}$ is the set of true boolean expressions over $\mathcal{F}$.

NFTA with ϵ-rules

Like in the word case, it is convenient to allow ϵ-moves in the reduction of a ground term by an automaton, i.e. the current state is changed but no new symbol of the term is processed. This is done by introducing a new type of rules in the set of transition rules of an automaton. A ***NFTA with ϵ-rules*** is like a NFTA except that now the set of transition rules contains ground transition rules of the form $f(q_1, \dots, q_n) \to q$, and ϵ-rules of the form $q \to q'$. The ability to make ϵ-moves does not allow the NFTA to accept non recognizable sets. But NFTA with ϵ-rules are useful in some constructions and simplify some proofs.

Example 5. Let $\mathcal{F} = \{cons(,), s(), 0, nil\}$. Consider the automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = \{q_{\mathsf{Nat}}, q_{\mathsf{List}}, q_{\mathsf{List}*}\}$, $Q_f = \{q_{\mathsf{List}}\}$, and $\Delta =$

$$\begin{array}{lrclrcl} \{ & 0 & \rightarrow & q_{\mathsf{Nat}} & s(q_{\mathsf{Nat}}) & \rightarrow & q_{\mathsf{Nat}} \\ & nil & \rightarrow & q_{\mathsf{List}} & cons(q_{\mathsf{Nat}}, q_{\mathsf{List}}) & \rightarrow & q_{\mathsf{List}*} \\ & q_{\mathsf{List}*} & \rightarrow & q_{\mathsf{List}}\}. & & & \end{array}$$

The recognized tree language is the set of Lisp-like lists of integers. If the final state set is $Q_f = \{q_{\mathsf{List}*}\}$, then the recognized tree language is the set of non empty Lisp-like lists of integers. The ϵ-rule $q_{\mathsf{List}*} \rightarrow q_{\mathsf{List}}$ says that a non empty list is a list. The reader should recognize the definition of an order-sorted algebra with the sorts Nat, List, and List$*$ (which stands for the non empty lists), and the inclusion List$* \subseteq$ List (see Section 3.4.1).

Theorem 1 (The equivalence of NFTA's with and without ϵ-rules). *If L is recognized by a NFTA with ϵ-rules, then L is recognized by a NFTA without ϵ-rules.*

Proof. Let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ be a NFTA with ϵ-rules. Consider the subset Δ_ϵ consisting of those ϵ-rules in Δ. We denote by *ϵ-closure(q)* the set of all states q' in Q such that there is a reduction of q into q' using rules in Δ_ϵ. The computation of such a set is equivalent to the question of what vertices can be reached from a given vertex in a directed graph. This can be done in quadratic time. Now let us define the NFTA $\mathcal{A}' = (Q, \mathcal{F}, Q_f, \Delta')$ where Δ' is defined by:

$$f(q_1, \dots, q_n) \rightarrow q' \in \Delta' \text{ iff } f(q_1, \dots, q_n) \rightarrow q \in \Delta, q' \in \epsilon\textit{-closure(q)}.$$

The proof of equivalence is an easy induction on the length of reductions. $\square$

Unless it is stated otherwise, we will now consider NFTA without ϵ-rules.

Deterministic Finite Tree Automata

Our definition of tree automata corresponds to the notion of nondeterministic finite tree automata. We will now define deterministic tree automata (DFTA) which are a special case of NFTA. It will turn out that, like in the word case, any language recognized by a NFTA can also be recognized by a DFTA. However, the NFTA are useful in proving theorems in tree language theory.

A tree automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ is ***deterministic*** (DFTA) if there are no two rules with the same left-hand side (and no ϵ-rule). Given a DFTA, there is at most one run for every ground term, i.e. for every ground term t, there is at most one state q such that $t \xrightarrow[\mathcal{A}]{*} q$. The latter property could be considered as a definition of deterministic tree automata, but the reader should note that it is not equivalent to the former one because the property could be satisfied even if there are two rules with the same left-hand side if some states are not accessible (see Example 6).

It is also useful to consider tree automata such that there is at least one run for every ground term. This leads to the following definition. A NFTA $\mathcal{A}$ is ***complete*** if there is at least one rule $f(q_1, \dots, q_n) \rightarrow q \in \Delta$ for all $n \geq 0$, $f \in \mathcal{F}_n$, and $q_1, \dots, q_n \in Q$. Let us note that for a complete DFTA there is exactly one run for every ground term. Lastly, for practical reasons, it is usual to consider automata in which unnecessary states are eliminated. A state q is ***accessible*** if there exists a ground term t such that $t \xrightarrow[\mathcal{A}]{*} q$. A NFTA $\mathcal{A}$ is said to be ***reduced*** if all its states are accessible.

Example 6.

The automaton defined in Example 3 is reduced, not complete, and it is not deterministic because there are two rules of left-hand side $g(q(x))$. Let us also note (see Example 3) that at least two runs (one is successful) can be defined on the term $g(g(f(g(a), a)))$.

The automaton defined in Example 4 is a complete and reduced DFTA.

Let $\mathcal{F} = \{g(), a\}$. Consider the automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = \{q_0, q_1, q\}$, $Q_f = \{q_0\}$, and Δ is the following set of transition rules:

$$\begin{array}{rrclrcl} \{ & a & \rightarrow & q_0 & g(q_0) & \rightarrow & q_1 \\ & g(q_1) & \rightarrow & q_0 & g(q) & \rightarrow & q_0 \\ & g(q) & \rightarrow & q_1\}. & & & \end{array}$$

This automaton is not deterministic because there are two rules of left-hand side $g(q)$, it is not reduced because state q is not accessible. Nevertheless, one should note that there is at most one run for every ground term t.

Let $\mathcal{F} = \{f(,), g(), a\}$. Consider the automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined in Example 2 by: $Q = \{q_a, q_g, q_f\}$, $Q_f = \{q_f\}$, and Δ is the following set of transition rules:

$$\begin{array}{rrclrcll} \{ & a & \rightarrow & q_a & g(q_a) & \rightarrow & q_g & \\ & g(q_g) & \rightarrow & q_g & f(q_g, q_g) & \rightarrow & q_f & \}. \end{array}$$

This automaton is deterministic and reduced. It is not complete because, for instance, there is no transition rule of left-hand side $f(q_a, q_a)$. It is easy to define a deterministic and complete automaton $\mathcal{A}'$ recognizing the same language by adding a "dead state". The automaton $\mathcal{A}' = (Q', \mathcal{F}, Q_f, \Delta')$ is defined by: $Q' = Q \cup \{\pi\}$, $\Delta' = \Delta \cup$

$$\begin{array}{rrclrcll} \{ & g(q_f) & \rightarrow & \pi & g(\pi) & \rightarrow & \pi & \\ & f(q_a, q_a) & \rightarrow & \pi & f(q_a, q_g) & \rightarrow & \pi & \\ & & \ldots & & f(\pi, \pi) & \rightarrow & \pi & \}. \end{array}$$

It is easy to generalize the construction given in Example 6 of a complete NFTA equivalent to a given NFTA: add a "dead state" π and all transition rules with right-hand side π such that the automaton is complete. The reader should note that this construction could be expensive because it may require $O(|\mathcal{F}| \times |Q|^{Arity(\mathcal{F})})$ new rules where $Arity(\mathcal{F})$ is the maximal arity of symbols in $\mathcal{F}$. Therefore we have the following:

Theorem 2. *Let L be a recognizable set of ground terms. Then there exists a complete finite tree automaton that accepts L.*

We now give an algorithm which outputs a reduced NFTA equivalent to a given NFTA as input. The main loop of this algorithm computes the set of accessible states.

```
**Reduction Algorithm RED**
**input:** NFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$
**begin**
    Set *Marked* to $\emptyset$ /* *Marked* is the set of accessible states */
    **repeat**
        Set *Marked* to *Marked* $\cup \{q\}$
        **where**
            $f \in \mathcal{F}_n$, $q_1, \ldots, q_n \in$ *Marked*, $f(q_1, \ldots, q_n) \to q \in \Delta$
    **until** no state can be added to *Marked*
    Set $Q_r$ to *Marked*
    Set $Q_{r_f}$ to $Q_f \cap$ *Marked*
    Set $\Delta_r$ to $\{f(q_1, \ldots, q_n) \to q \in \Delta \mid q, q_1, \ldots, q_n \in \mathit{Marked}\}$
    **output:** DFTA $\mathcal{A}_r = (Q_r, \mathcal{F}, Q_{r_f}, \Delta_r)$
**end**
```

Obviously all states in the set *Marked* are accessible, and an easy induction shows that all accessible states are in the set *Marked*. And, the NFTA $\mathcal{A}_r$ accepts the tree language $L(\mathcal{A})$. Consequently we have:

Theorem 3. *Let L be a recognizable set of ground terms. Then there exists a reduced finite tree automaton that accepts L.*

Now, we consider the reduction of nondeterminism. Since every DFTA is a NFTA, it is clear that the class of recognizable languages includes the class of languages accepted by DFTA's. However it turns out that these classes are equal. We prove that, for every NFTA, we can construct an equivalent DFTA. The proof is similar to the proof of equivalence between DFA's and NFA's in the word case. The proof is based on the "subset construction". Consequently, the number of states of the equivalent DFTA can be exponential in the number of states of the given NFTA (see Example 8). But, in practice, it often turns out that many states are not accessible. Therefore, we will present in the proof of the following theorem a construction of a DFTA where only the accessible states are considered, i.e. the given algorithm outputs an equivalent and reduced DFTA from a given NFTA as input.

Theorem 4 (The equivalence of DFTA's and NFTA's). *Let L be a recognizable set of ground terms. Then there exists a DFTA that accepts L.*

Proof. First, we give a theoretical construction of a DFTA equivalent to a NFTA. Let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ be a NFTA. Define a DFTA $\mathcal{A}_d = (Q_d, \mathcal{F}, Q_{d_f}, \Delta_d)$, as follows. The states of Q_d are all the subsets of the state set Q of $\mathcal{A}$. That is, $Q_d = 2^Q$. We denote by s a state of Q_d, i.e. $s = \{q_1, \ldots, q_n\}$ for some states $q_1, \ldots, q_n \in Q$. We define

$$f(s_1, \ldots, s_n) \to s \in \Delta_d$$
$$\text{iff}$$
$$s = \{q \in Q \mid \exists q_1 \in s_1, \ldots, \exists q_n \in s_n, f(q_1, \ldots, q_n) \to q \in \Delta\}.$$

And Q_{d_f} is the set of all states in Q_d containing a final state of $\mathcal{A}$.

We now give an algorithmic construction where only the accessible states are considered.

```
Determinization Algorithm DET
input: NFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$
begin
    /* A state $s$ of the equivalent DFTA is in $2^Q$ */
    Set $Q_d$ to $\emptyset$; set $\Delta_d$ to $\emptyset$
    repeat
        Set $Q_d$ to $Q_d \cup \{s\}$; Set $\Delta_d$ to $\Delta_d \cup \{f(s_1, \dots, s_n) \to s\}$
        where
            $f \in \mathcal{F}_n$, $s_1, \dots, s_n \in Q_d$,
            $s = \{q \in Q \mid \exists q_1 \in s_1, \dots, q_n \in s_n, f(q_1, \dots, q_n) \to q \in \Delta\}$
    until no rule can be added to $\Delta$
    Set $Q_{d_f}$ to $\{s \in Q_d \mid s \cap Q_f \neq \emptyset\}$
    output: DFTA $\mathcal{A}_d = (Q_d, \mathcal{F}, Q_{d_f}, \Delta_d)$
end
```

It is immediate from the definition of the determinization algorithm that $\mathcal{A}_d$ is a deterministic and reduced tree automaton. In order to prove that $L(\mathcal{A}) = L(\mathcal{A}_d)$, we now prove that:

$$(t \xrightarrow[\mathcal{A}_d]{*} s) \text{ iff } (s = \{q \in Q \mid t \xrightarrow[\mathcal{A}]{*} q\}).$$

The proof is an easy induction on the structure of terms.

- base case: let us consider $t = a \in \mathcal{F}_0$. Then, there is only one rule $a \to s$ in Δ_d where $s = \{q \in Q \mid a \to q \in \Delta\}$. Consequently, the base case is straightforward.
- induction step: let us consider a term $t = f(t_1, \dots, t_n)$. First, let us suppose that $t \xrightarrow[\mathcal{A}_d]{*} f(s_1, \dots, s_n) \to_{\mathcal{A}_d} s$. By induction hypothesis, we have $s_i = \{q \in Q \mid t_i \xrightarrow[\mathcal{A}]{*} q\}$, for each $i \in \{1, \dots, n\}$. By construction of the rule $f(s_1, \dots, s_n) \to s \in \Delta_d$ in the determinization algorithm, it is immediate that $s = \{q \in Q \mid t \xrightarrow[\mathcal{A}]{*} q\}$. Second, if $s = \{q \in Q \mid t \xrightarrow[\mathcal{A}]{*} q\}$, it is easy to prove that $t \xrightarrow[\mathcal{A}_d]{*} s$.

□

Example 7. Let $\mathcal{F} = \{f(,), g(), a\}$. Consider the automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined in Example 3 by: $Q = \{q, q_g, q_f\}$, $Q_f = \{q_f\}$, and $\Delta =$

$$\begin{array}{lrcllrcl}
\{ & a & \to & q & & g(q) & \to & q \\
 & g(q) & \to & q_g & & g(q_g) & \to & q_f \\
 & f(q,q) & \to & q & \}. & & &
\end{array}$$

Given $\mathcal{A}$ as input, the determinization algorithm outputs the DFTA $\mathcal{A}_d = (Q_d, \mathcal{F}, Q_{d_f}, \Delta_d)$ defined by: $Q_d = \{\{q\}, \{q, q_g\}, \{q, q_g, q_f\}\}$, $Q_{d_f} = \{\{q, q_g, q_f\}\}$, and $\Delta_d =$

$$\begin{array}{rrcll}
\{ & a & \rightarrow & \{q\} & \\
 & g(\{q\}) & \rightarrow & \{q, q_g\} & \\
 & g(\{q, q_g\}) & \rightarrow & \{q, q_g, q_f\} & \\
 & g(\{q, q_g, q_f\}) & \rightarrow & \{q, q_g, q_f\} & \} \\
\cup \; \{ & f(s_1, s_2) & \rightarrow & \{q\} & \mid \quad s_1, s_2 \in Q_d \quad \}.
\end{array}$$

We now give an example where an exponential blow-up occurs in the determinization process. This example is the same used in the word case.

Example 8. Let $\mathcal{F} = \{f(), g(), a\}$ and let n be an integer. And let us consider the tree language

$$L = \{t \in T(\mathcal{F}) \mid \text{the symbol at position } 1^n \text{ is } f\}.$$

Let us consider the NFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = \{q, q_1, \ldots, q_{n+1}\}$, $Q_f = \{q_{n+1}\}$, and $\Delta =$

$$\begin{array}{rrclrcll}
\{ & a & \rightarrow & q & f(q) & \rightarrow & q & \\
 & g(q) & \rightarrow & q & f(q) & \rightarrow & q_1 & \\
 & g(q_1) & \rightarrow & q_2 & f(q_1) & \rightarrow & q_2 & \\
 & \ldots & & & & & & \\
 & g(q_n) & \rightarrow & q_{n+1} & f(q_n) & \rightarrow & q_{n+1} & \}.
\end{array}$$

The NFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ accepts the tree language L, and it has $n + 2$ states. Using the subset construction, the equivalent DFTA $\mathcal{A}_d$ has 2^{n+1} states. Any equivalent automaton has to memorize the $n + 1$ last symbols of the input tree. Therefore, it can be proved that any DFTA accepting L has at least 2^{n+1} states. The automaton $\mathcal{A}_d$ is minimal in the number of states (minimal tree automata are defined in Section 1.5).

If a finite tree automaton is deterministic, we can replace the transition relation Δ by a transition function δ. Therefore, it is sometimes convenient to consider a DFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \delta)$ where

$$\delta : \bigcup_n \mathcal{F}_n \times Q^n \rightarrow Q \; .$$

The computation of such an automaton on a term t as input tree can be viewed as an evaluation of t on finite domain Q. Indeed, define the labeling function $\hat{\delta} : T(\mathcal{F}) \rightarrow Q$ inductively by

$$\hat{\delta}(f(t_1, \ldots, t_n)) = \delta(f, \hat{\delta}(t_1), \ldots, \hat{\delta}(t_n)) \; .$$

We shall for convenience confuse δ and $\hat{\delta}$.

We now make clear the connections between our definitions and the language theoretical definitions of tree automata and of recognizable tree languages. Indeed, the reader should note that a DFTA is just a finite $\mathcal{F}$-algebra $\mathcal{A}$ consisting of a finite carrier $|\mathcal{A}| = Q$ and a distinguished n-ary function $f^{\mathcal{A}} : Q^n \rightarrow Q$ for each n-ary symbol $f \in \mathcal{F}$ together with a specified subset Q_f of Q. A ground term t is accepted by $\mathcal{A}$ if $\delta(t) = q \in Q_f$ where δ is the unique $\mathcal{F}$-algebra homomorphism $\delta : T(\mathcal{F}) \to \mathcal{A}$.

Example 9. Let $\mathcal{F} = \{f(,), a\}$ and consider the $\mathcal{F}$-algebra $\mathcal{A}$ with $|\mathcal{A}| = Q = Z_2 = \{0, 1\}$, $f^{\mathcal{A}} = +$ where the sum is formed modulo 2, $a^{\mathcal{A}} = 1$, and let $Q_f = \{0\}$. $\mathcal{A}$ and Q_f defines a DFTA. The recognized tree language is the set of ground terms over $\mathcal{F}$ with an even number of leaves.

Since DFTA and NFTA accept the same sets of tree languages, we shall not distinguish between them unless it becomes necessary, but shall simply refer to both as tree automata (FTA).

1.2 The pumping Lemma for Recognizable Tree Languages

We now give an example of a tree language which is not recognizable.

Example 10. Let $\mathcal{F} = \{f(,), g(), a\}$. Let us consider the tree language $L = \{f(g^i(a), g^i(a)) \mid i > 0\}$. Let us suppose that L is recognizable by an automaton $\mathcal{A}$ having k states. Now, consider the term $t = f(g^k(a), g^k(a))$. t belongs to L, therefore there is a successful run of $\mathcal{A}$ on t. As k is the cardinality of the state set, there are two distinct positions along the first branch of the term labeled with the same state. Therefore, one could cut the first branch between these two positions leading to a term $t' = f(g^j(a), g^k(a))$ with $j < k$ such that a successful run of $\mathcal{A}$ can be defined on t'. This leads to a contradiction with $L(\mathcal{A}) = L$.

This (sketch of) proof can be generalized by proving a ***pumping lemma*** for recognizable tree languages. This lemma is extremely useful in proving that certain sets of ground terms are not recognizable. It is also useful for solving decision problems like emptiness and finiteness of a recognizable tree language (see Section 1.7).

Pumping Lemma. *Let L be a recognizable set of ground terms. Then, there exists a constant $k > 0$ satisfying: for every ground term t in L such that $\mathcal{H}eight(t) > k$, there exist a context $C \in \mathcal{C}(\mathcal{F})$, a non trivial context $C' \in \mathcal{C}(\mathcal{F})$, and a ground term u such that $t = C[C'[u]]$ and, for all $n \geq 0$ $C[C'^n[u]] \in L$.*

Proof. Let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ be a FTA such that $L = L(\mathcal{A})$ and let $k = |Q|$ be the cardinality of the state set Q. Let us consider a ground term t in L such that $\mathcal{H}eight(t) > k$ and consider a successful run r of $\mathcal{A}$ on t. Now let us consider a path in t of length strictly greater than k. As k is defined to be the cardinality of the state set Q, there are two positions $p_1 < p_2$ along this path such that $r(p_1) = r(p_2) = q$ for some state q. Let u be the ground subterm of t at position p_2. Let u' be the ground subterm of t at position p_1, there exists a non trivial context C' such that $u' = C'[u]$. Now define the context C such that

$t = C[C'[u]]$. Consider a term $C[C'^n[u]]$ for some integer $n > 1$, a successful run can be defined on this term. Indeed suppose that r corresponds to the reduction $t \xrightarrow[\mathcal{A}]{*} q_f$ where q_f is a final state of $\mathcal{A}$, then we have:

$$C[C'^n[u]] \xrightarrow[\mathcal{A}]{*} C[C'^n[q]] \xrightarrow[\mathcal{A}]{*} C[C'^{n-1}[q]] \ldots \xrightarrow[\mathcal{A}]{*} C[q] \xrightarrow[\mathcal{A}]{*} q_f.$$

The same holds when $n = 0$. □

Example 11. Let $\mathcal{F} = \{f(,), a\}$. Let us consider the tree language $L = \{t \in T(\mathcal{F}) \mid |\mathcal{P}os(t)|$ is a prime number$\}$. We can prove that L is not recognizable. For all $k > 0$, consider a term t in L whose height is greater than k. For all contexts C, non trivial contexts C', and terms u such that $t = C[C'[u]]$, there exists n such that $C[C'^n[u]] \notin L$.

From the Pumping Lemma, it is immediate to derive the following corollary:

Corollary 1. *Let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ be a FTA. Then $L(\mathcal{A})$ is non empty if and only there exists a term t in $L(\mathcal{A})$ with $\mathcal{H}eight(t) \leq |Q|$. Then $L(\mathcal{A})$ is infinite if and only if there exists a term t in $L(\mathcal{A})$ with $|Q| < \mathcal{H}eight(t) \leq 2 \times |Q|$.*

1.3 Closure Properties of Recognizable Tree Languages

A ***closure property*** of a class of (tree) languages is the fact that the class is closed under a particular operation. We are interested in effective closure properties where, given representations for languages in the class, there is an algorithm to construct a representation for the language that results by applying the operation to these languages. Let us note that the equivalence between NFTA and DFTA is effective, thus we may choose the representation that suits us best. Nevertheless, the determinization algorithm may output a DFTA whose number of states is exponential in the number of states of the given NFTA. For the different closure properties, we give effective constructions and we give the properties of the resulting FTA depending on the properties of the given FTA as input. In this section, we consider the Boolean set operations: union, intersection, and complementation. Other operations will be studied in the next sections. Complexity results are given in Section 1.7.

Theorem 5. *The class of recognizable tree languages is closed under union, under complementation, and under intersection.*

Union

Let L_1 and L_2 be two recognizable tree languages. Thus there are tree automata $\mathcal{A}_1 = (Q_1, \mathcal{F}, Q_{f1}, \Delta_1)$ and $\mathcal{A}_2 = (Q_2, \mathcal{F}, Q_{f2}, \Delta_2)$ with $L_1 = L(\mathcal{A}_1)$ and $L_2 = L(\mathcal{A}_2)$. Since we may rename states of a tree automaton, without loss of generality, we may suppose that $Q_1 \cap Q_2 = \emptyset$. Now, let us consider the FTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = Q_1 \cup Q_2$, $Q_f = Q_{f1} \cup Q_{f2}$, and

$\Delta = \Delta_1 \cup \Delta_2$. The equality between $L(\mathcal{A})$ and $L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$ is straightforward. Let us note that $\mathcal{A}$ is nondeterministic and not complete, even if $\mathcal{A}_1$ and $\mathcal{A}_2$ are deterministic and complete.

We now give another construction which preserves determinism. The intuitive idea is to process in parallel a term by the two automata. For this we consider a product automaton. Let us suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are complete. And, let us consider the FTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = Q_1 \times Q_2$, $Q_f = Q_{f1} \times Q_2 \cup Q_1 \times Q_{f2}$, and $\Delta = \Delta_1 \times \Delta_2$ where

$$\begin{aligned}\Delta_1 \times \Delta_2 = \{ f((q_1, q_1'), \ldots, f(q_n, q_n')) \rightarrow (q, q') \mid \\ f(q_1, \ldots, q_n) \rightarrow q \in \Delta_1 \; f({q'}_1, \ldots, {q'}_n) \rightarrow q' \in \Delta_2 \}\end{aligned}$$

It is easy to prove that $L(\mathcal{A}) = L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$. The reader should note that the hypothesis that the two given tree automata are complete is crucial in the proof. Indeed, suppose for instance that a ground term t is accepted by $\mathcal{A}_1$ but not by $\mathcal{A}_2$. Moreover suppose that $\mathcal{A}_2$ is not complete and that there is no run of $\mathcal{A}_2$ on t, then the product automaton does not accept t because there is no run of the product automaton on t. The reader should also note that the construction preserves determinism, i.e. if the two given automata are deterministic, then the product automaton is also deterministic.

Complementation

Let L be a recognizable tree language. Let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ be a complete DFTA such that $L(\mathcal{A}) = L$. Now, complement the final state set to recognize the complement of L. That is, let $\mathcal{A}^c = (Q, \mathcal{F}, Q_f^c, \Delta)$ with $Q_f^c = Q - Q_f$, the DFTA $\mathcal{A}^c$ recognizes the tree language $T(\mathcal{F}) - L$. If you are given with a NFTA, first, it is necessary to apply the determinization algorithm, and second complement the final state set. This could lead to an exponential blow-up.

Intersection

Closure under intersection follows from closure under union and complementation because

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}.$$

where we denote by $\overline{L}$ the complement $T(\mathcal{F}) - L$ of set L. But if the recognizable tree languages are defined by NFTA, we have to use the complementation construction, therefore the determinization process is used leading to an exponential blow-up. Consequently, we now give a direct construction which does not use the determinization algorithm. Let $\mathcal{A}_1 = (Q_1, \mathcal{F}, Q_{f1}, \Delta_1)$ and $\mathcal{A}_2 = (Q_2, \mathcal{F}, Q_{f2}, \Delta_2)$ be FTA such that $L(\mathcal{A}_1) = L_1$ and $L(\mathcal{A}_2) = L_2$. And, consider the FTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = Q_1 \times Q_2$, $Q_f = Q_{f1} \times Q_{f2}$, and $\Delta = \Delta_1 \times \Delta_2$. $\mathcal{A}$ recognizes $L_1 \cap L_2$. Moreover the reader should note that $\mathcal{A}$ is deterministic if $\mathcal{A}_1$ and $\mathcal{A}_2$ are deterministic.

1.4 Tree homomorphisms

We now consider tree transformations and study the closure properties under these tree transformations. In this section we are interested with tree transformations preserving the structure of trees. Thus, we restrict ourselves to tree homomorphisms. Tree homomorphisms are a generalization of homomorphisms for words (considered as unary terms) to the case of arbitrary ranked alphabets. In the word case, it is known that the class of regular sets is closed under homomorphisms and inverse homomorphisms. The situation is different in the tree case because if recognizable tree languages are closed under inverse homomorphisms, they are closed only under a subclass of homomorphisms, i.e. linear homomorphisms (duplication of terms is forbidden). First, we define tree homomorphisms.

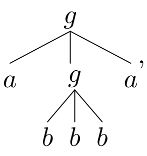
Let $\mathcal{F}$ and $\mathcal{F}'$ be two sets of function symbols, possibly not disjoint. For each $n > 0$ such that $\mathcal{F}$ contains a symbol of arity n, we define a set of variables $\mathcal{X}_n = \{x_1, \dots, x_n\}$ disjoint from $\mathcal{F}$ and $\mathcal{F}'$.

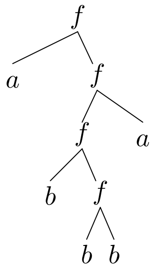
Let $h_\mathcal{F}$ be a mapping which, with $f \in \mathcal{F}$ of arity n, associates a term $t_f \in T(\mathcal{F}', \mathcal{X}_n)$. The ***tree homomorphism*** $h : T(\mathcal{F}) \to T(\mathcal{F}')$ determined by $h_\mathcal{F}$ is defined as follows:

- $h(a) = t_a \in T(\mathcal{F}')$ for each $a \in \mathcal{F}$ of arity 0,
- $h(f(t_1, \dots, t_n)) = t_f\{x_1 \leftarrow h(t_1), \dots, x_n \leftarrow h(t_n)\}$

where $t_f\{x_1 \leftarrow h(t_1), \dots, x_n \leftarrow h(t_n)\}$ is the result of applying the substitution $\{x_1 \leftarrow h(t_1), \dots, x_n \leftarrow h(t_n)\}$ to the term t_f.

Example 12. Let $\mathcal{F} = \{g(,,), a, b\}$ and $\mathcal{F}' = \{f(,), a, b\}$. Let us consider the tree homomorphism h determined by $h_\mathcal{F}$ defined by: $h_\mathcal{F}(g) = f(x_1, f(x_2, x_3))$, $h_\mathcal{F}(a) = a$ and $h_\mathcal{F}(b) = b$. For instance, we have:

If $t =$ 

, then $h(t) =$ 

This homomorphism can be used to transform ternary trees into binary trees. Let us now consider $\mathcal{F} = \{and(,), or(,), not(), 0, 1\}$ and $\mathcal{F}' = \{or(,), not(), 0, 1\}$. Let us consider the tree homomorphism h determined by $h_\mathcal{F}$ defined by: $h_\mathcal{F}(and) = not(or(not(x_1), not(x_2)))$, and $h_\mathcal{F}$ is the identity otherwise. This homomorphism transforms a boolean formula in an equivalent boolean formula which does not contain *and*.

A tree homomorphism is ***linear*** if for each $f \in \mathcal{F}$ of arity n, $h_\mathcal{F}(f) = t_f$ is a linear term in $T(\mathcal{F}', \mathcal{X}_n)$. The following example shows that tree homomorphisms do not always preserve recognizability.

Example 13. Let $\mathcal{F} = \{f(), g(), a\}$ and $\mathcal{F}' = \{f'(,), g(), a\}$. Let us consider the tree homomorphism h determined by $h_\mathcal{F}$ defined by: $h_\mathcal{F}(f) = f'(x_1, x_1)$, $h_\mathcal{F}(g) = g(x_1)$, and $h_\mathcal{F}(a) = a$. h is not linear. Let $L = \{f(g^i(a)) \mid i \geq 0\}$, then L is a recognizable tree language. $h(L) = \{f'(g^i(a), g^i(a)) \mid i \geq 0\}$ is not recognizable (see Example 10).

Theorem 6 (Linear homomorphisms preserve recognizability). *Let h be a linear tree homomorphism and L be a recognizable tree language, then $h(L)$ is a recognizable tree language.*

Proof. Let L be a recognizable tree language. Let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ be a reduced DFTA such that $L(\mathcal{A}) = L$. Let h be a linear tree homomorphism from $T(\mathcal{F})$ into $T(\mathcal{F}')$ determined by a mapping $h_{\mathcal{F}}$.

First, let us define a NFTA $\mathcal{A}' = (Q', \mathcal{F}', Q'_f, \Delta')$. Let us consider a rule $r = f(q_1, \ldots, q_n) \rightarrow q$ in Δ and consider the linear term $t_f = h_{\mathcal{F}}(f) \in T(\mathcal{F}', \mathcal{X}_n)$ and the set of positions $\mathcal{P}os(t_f)$. We define a set of states $Q^r = \{q^r_p \mid p \in \mathcal{P}os(t_f)\}$, and we define a set of rules Δ_r as follows: for all positions p in $\mathcal{P}os(t_f)$

- if $t_f(p) = g \in \mathcal{F}'_k$, then $g(q^r_{p1}, \ldots, q^r_{pk}) \rightarrow q^r_p \in \Delta_r$,
- if $t_f(p) = x_i$, then $q_i \rightarrow q^r_p \in \Delta_r$,
- $q^r_\epsilon \rightarrow q \in \Delta_r$.

The preceding construction is made for each rule in Δ. We suppose that all the state sets Q^r are disjoint and that they are disjoint from Q. Now define $\mathcal{A}'$ by:

- $Q' = Q \cup \bigcup_{r \in \Delta} Q^r$,
- $Q'_f = Q_f$,
- $\Delta' = \bigcup_{r \in \Delta} \Delta^r$.

Second, we have to prove that $h(L) = L(\mathcal{A}')$.

$h(L) \subseteq L(\mathcal{A}')$. We prove that if $t \xrightarrow[\mathcal{A}]{*} q$ then $h(t) \xrightarrow[\mathcal{A}']{*} q$ by induction on the length of the reduction of ground term $t \in T(\mathcal{F})$ by automaton $\mathcal{A}$.

- Base case. Suppose that $t \rightarrow_{\mathcal{A}} q$. Then $t = a \in \mathcal{F}_0$ and $a \rightarrow q \in \Delta$. Then, by definition of $\mathcal{A}'$, it is easy to prove that $h(a) = t_a \xrightarrow[\mathcal{A}']{*} q$.
- Induction step.
  Suppose that $t = f(u_1, \ldots, u_n)$, then $h(t) = t_f\{x_1 \leftarrow h(u_1), \ldots, x_n \leftarrow h(u_n)\}$. Moreover suppose that $t \xrightarrow[\mathcal{A}]{*} f(q_1, \ldots, q_n) \rightarrow_{\mathcal{A}} q$. By induction hypothesis, we have $h(u_i) \xrightarrow[\mathcal{A}']{*} q_i$, for each i in $\{1, \ldots, n\}$. Now by definition of $\mathcal{A}'$, it is easy to prove that $t_f\{x_1 \leftarrow q_1, \ldots, x_n \leftarrow q_n\} \xrightarrow[\mathcal{A}']{*} q$.

$h(L) \supseteq L(\mathcal{A}')$. We prove that if $t' \xrightarrow[\mathcal{A}']{*} q \in Q$ then $t' = h(t)$ with $t \xrightarrow[\mathcal{A}]{*} q$ for some $t \in T(\mathcal{F})$. The proof is by induction on the number of states in Q occurring along the reduction $t' \xrightarrow[\mathcal{A}']{*} q \in Q$.

- Base case. Suppose that $t' \xrightarrow[\mathcal{A}']{*} q \in Q$ and no state in Q apart from q occurs in the reduction. Then, because the state sets Q^r are disjoint, only rules of some Δ^r can be used in the reduction. Thus, t' is ground, $t' = h_{\mathcal{F}}(f)$ for some symbol $f \in \mathcal{F}$, and $r = f(q_1, \dots, q_n) \to q$. Because the automaton is reduced, there is some ground term t with $\mathcal{H}ead(t) = f$ such that $t' = h(t)$ and $t \xrightarrow[\mathcal{A}]{*} q$.
- Induction step. Suppose that
$$t' \xrightarrow[\mathcal{A}']{*} v\{x'_1 \leftarrow q_1, \dots, x'_m \leftarrow q_m\} \xrightarrow[\mathcal{A}']{*} q$$
where v is a linear term in $T(\mathcal{F}', \{x'_1, \dots, x'_m\})$, $t' = v\{x'_1 \leftarrow u'_1, \dots, x'_m \leftarrow u'_m\}$, $u'_i \xrightarrow[\mathcal{A}']{*} q_i \in Q$, and no state in Q apart from q occurs in the reduction of $v\{x'_1 \leftarrow q_1, \dots, x'_m \leftarrow q_m\}$ in q. The reader should note that different variables can be substituted by the same state. Then, because the state sets Q^r are disjoint, only rules of some Δ^r can be used in the reduction of $v\{x'_1 \leftarrow q_1, \dots, x'_m \leftarrow q_m\}$ in q. Thus, there exists some linear term t_f such that $v\{x'_1 \leftarrow q_1, \dots, x'_m \leftarrow q_m\} = t_f\{x_1 \leftarrow q_1, \dots, x_n \leftarrow q_n\}$ for some symbol $f \in \mathcal{F}_n$ and $r = f(q_1, \dots, q_n) \to q \in \Delta$. By induction hypothesis, there are terms $u_1, \dots, u_m$ in L such that $u'_i = h(u_i)$ and $u_i \xrightarrow[\mathcal{A}]{*} q_i$ for each i in $\{1, \dots, m\}$. Now consider the term $t = f(v_1, \dots, v_n)$, where $v_i = u_i$ if x_i occurs in t_f and v_i is some term such that $v_i \xrightarrow[\mathcal{A}]{*} q_i$ otherwise (such v_i always exist because $\mathcal{A}$ is reduced). We have $h(t) = t_f\{x_1 \leftarrow h(v_1), \dots, x_n \leftarrow h(v_n)\}$, $h(t) = v\{x'_1 \leftarrow h(u_1), \dots, x'_m \leftarrow h(u_m)\}$, $h(t) = t'$. Moreover, by definition of the v_i and by induction hypothesis, we have $t \xrightarrow[\mathcal{A}]{*} q$. Note that if q_i occurs more than once, you can substitute q_i by any term satisfying the conditions. The proof does not work for the non linear case because you have to check that different occurrences of some state q_i corresponding to the same variable $x_j \in \mathcal{V}ar(t_f)$ can only be substituted by equal terms.

□

Only linear tree homomorphisms preserve recognizability. But, now we show that arbitrary inverse homomorphisms preserve recognizability.

Theorem 7 (Inverse homomorphisms preserve recognizability). *Let h be a tree homomorphism and L be a recognizable tree language, then $h^{-1}(L)$ is a recognizable tree language.*

Proof. Let h be a tree homomorphism from $T(\mathcal{F})$ into $T(\mathcal{F}')$ determined by a mapping $h_{\mathcal{F}}$. Let $\mathcal{A}' = (Q', \mathcal{F}', Q'_f, \Delta')$ be a complete DFTA such that $L(\mathcal{A}') = L$. We define a DFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ by $Q = Q'$, $Q_f = Q'_f$ and Δ is defined by the following:

$$\text{if } t_f\{x_1 \leftarrow q_1, \dots, x_n \leftarrow q_n\} \xrightarrow[\mathcal{A}']{*} q \text{ then } f(q_1, \dots, q_n) \to q \in \Delta.$$

It is obvious that Δ is computable. It is easy to show by induction on the structure of terms that $t \xrightarrow[\mathcal{A}]{*} q$ if and only if $h(t) \xrightarrow[\mathcal{A}']{*} q$. □

It can be proved that the class of recognizable tree languages is the smallest class of tree languages closed by linear tree homomorphisms and inverse tree homomorphisms. Tree homomorphisms do not in general preserve recognizability, therefore let us consider the following problem: given as instance a recognizable tree language L and a tree homomorphism h, is the set $h(L)$ recognizable ? To our knowledge it is not known whether this problem is decidable. The reader should note that if this problem is decidable, the problem whether the set of normal forms of a rewrite system is recognizable is easily shown decidable (see Exercises 6 and 10).

As a conclusion we consider different special types of tree homomorphisms. These homomorphisms will be used in the next sections in order to simplify some proofs and will be useful in Chapter 6. Let h be a tree homomorphism determined by $h_{\mathcal{F}}$. The tree homomorphism h is said to be:

- ***ϵ-free*** if for each symbol $f \in \mathcal{F}$, t_f is not reduced to a variable.
- ***symbol to symbol*** if for each symbol $f \in \mathcal{F}$, $\mathcal{H}eight(t_f) = 1$. The reader should note that with our definitions a symbol to symbol tree homomorphism is ϵ-free. A linear symbol to symbol tree homomorphism changes the label of the input symbol, possibly erases some subtrees and possibly modifies order of subtrees.
- ***complete*** if for each symbol $f \in \mathcal{F}_n$, $\mathcal{V}ar(t_f) = \mathcal{X}_n$.
- a ***delabeling*** if h is a complete, linear, symbol to symbol tree homomorphism. Such a delabeling only changes the label of the input symbol and possibly order of subtrees.
- ***alphabetic*** if for each symbol $f \in \mathcal{F}_n$, $t_f = g(x_1, \dots, x_n)$, where $g \in \mathcal{F}'_n$.

As a corollary of Theorem 6, alphabetic tree homomorphisms, delabelings and linear, symbol to symbol tree homomorphisms preserve recognizability. It is easy to prove that for these classes of tree homomorphisms, given h and a FTA $\mathcal{A}$ such that $L(\mathcal{A}) = L$ as instance, a FTA for the recognizable tree language $h(L)$ can be constructed in linear time. The same holds for $h^{-1}(L)$.

Example 14. Let $\mathcal{F} = \{f(,), g(), a\}$ and $\mathcal{F}' = \{f'(,), g'(), a'\}$. Let us consider some tree homomorphisms h determined by different $h_{\mathcal{F}}$.

- $h_{\mathcal{F}}(f) = x_1$, $h_{\mathcal{F}}(g) = f'(x_1, x_1)$, and $h_{\mathcal{F}}(a) = a'$. h is not linear, not ϵ-free, and not complete.
- $h_{\mathcal{F}}(f) = g'(x_1)$, $h_{\mathcal{F}}(g) = f'(x_1, x_1)$, and $h_{\mathcal{F}}(a) = a'$. h is a non linear symbol to symbol tree homomorphism. h is not complete.
- $h_{\mathcal{F}}(f) = f'(x_2, x_1)$, $h_{\mathcal{F}}(g) = g'(x_1)$, and $h_{\mathcal{F}}(a) = a'$. h is a delabeling.
- $h_{\mathcal{F}}(f) = f'(x_1, x_2)$, $h_{\mathcal{F}}(g) = g'(x_1)$, and $h_{\mathcal{F}}(a) = a'$. h is an alphabetic tree homomorphism.

1.5 Minimizing Tree Automata

In this section, we prove that, like in the word case, there exists a unique minimal automaton in the number of states for a given recognizable tree language.

A Myhill-Nerode Theorem for Tree Languages

The ***Myhill-Nerode Theorem*** is a classical result in the theory of finite automata. This theorem gives a characterization of the recognizable sets and it has numerous applications. A consequence of this theorem, among other consequences, is that there is essentially a unique minimum state DFA for every recognizable language over finite alphabet. The Myhill-Nerode Theorem generalizes in a straightforward way to automata on finite trees.

An equivalence relation $\equiv$ on $T(\mathcal{F})$ is a ***congruence*** on $T(\mathcal{F})$ if for every $f \in \mathcal{F}_n$

$$u_i \equiv v_i \; 1 \leq i \leq n \Rightarrow f(u_1, \ldots, u_n) \equiv f(v_1, \ldots, v_n) \; .$$

It is of ***finite index*** if there are only finitely many $\equiv$-classes. Equivalently a congruence is an equivalence relation closed under context, i.e. for all contexts $C \in \mathcal{C}(\mathcal{F})$, if $u \equiv v$, then $C[u] \equiv C[v]$. For a given tree language L, let us define the congruence $\equiv_L$ on $T(\mathcal{F})$ by: $u \equiv_L v$ if for all contexts $C \in \mathcal{C}(\mathcal{F})$,

$$C[u] \in L \text{ iff } C[v] \in L.$$

We are now ready to give the Theorem:

Myhill-Nerode Theorem. *The following three statements are equivalent:*

(i) L *is a recognizable tree language*

(ii) L *is the union of some equivalence classes of a congruence of finite index*

(iii) the relation $\equiv_L$ *is a congruence of finite index.*

Proof.

- *(i) ⇒ (ii)* Assume that L is recognized by some complete DFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \delta)$. We consider δ as a transition function. Let us consider the relation $\equiv_{\mathcal{A}}$ defined on $T(\mathcal{F})$ by: $u \equiv_{\mathcal{A}} v$ if $\delta(u) = \delta(v)$. Clearly $\equiv_{\mathcal{A}}$ is a congruence relation and it is of finite index, since the number of equivalence classes is at most the number of states in Q. Furthermore, L is the union of those equivalence classes that include a term u such that $\delta(u)$ is a final state.

- *(ii) ⇒ (iii)* Let us denote by $\sim$ the congruence of finite index. And let us assume that $u \sim v$. By an easy induction on the structure of terms, it can be proved that $C[u] \sim C[v]$ for all contexts $C \in \mathcal{C}(\mathcal{F})$. Now, L is the union of some equivalence classes of $\sim$, thus we have $C[u] \in L$ iff $C[v] \in L$. Thus $u \equiv_L v$, and the equivalence class of u in $\sim$ is contained in the equivalence class of u in $\equiv_L$. Consequently, the index of $\equiv_L$ is lower or equal than the index of $\sim$ which is finite.

- *(iii) ⇒ (i)* Let Q_{min} be the finite set of equivalence classes of $\equiv_L$. And let us denote by $[u]$ the equivalence class of a term u. Let the transition function δ_{min} be defined by:

  $$\delta_{min}(f([u_1], \dots, [u_n])) = [f(u_1, \dots, u_n)].$$

  The definition of δ_{min} is consistent because $\equiv_L$ is a congruence. And let $Q_{min_f} = \{[u] \mid u \in L\}$. The DFTA $\mathcal{A}_{min} = (Q_{min}, \mathcal{F}, Q_{min_f}, \delta_{min})$ recognizes the tree language L.

□

As a corollary of the Myhill-Nerode Theorem, we can deduce an other algebraic characterization of recognizable tree languages. This characterization is a reformulation of the definition of recognizability. A set of ground terms L is recognizable if and only if there exist a finite $\mathcal{F}$-algebra $\mathcal{A}$, a $\mathcal{F}$-algebra homomorphism $\phi : T(\mathcal{F}) \to \mathcal{A}$ and a subset A' of the carrier $|\mathcal{A}|$ of $\mathcal{A}$ such that $L = \phi^{-1}(A')$.

Minimization of Tree Automata

First, we prove the existence and uniqueness of the minimum DFTA for a recognizable tree language. It is a consequence of the Myhill-Nerode Theorem because of the following result:

Corollary 2. *The minimum DFTA recognizing a recognizable tree language L is unique up to a renaming of the states and is given by $\mathcal{A}_{min}$ in the proof of Myhill-Nerode Theorem.*

Proof. Assume that L is recognized by some DFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \delta)$. The relation $\equiv_{\mathcal{A}}$ is a refinement of $\equiv_L$ (see the proof of Myhill-Nerode Theorem). Therefore the number of states of $\mathcal{A}$ is greater than or equal to the number of states of $\mathcal{A}_{min}$. If equality holds, $\mathcal{A}$ is reduced, i.e. all states are accessible, because otherwise a state could be removed leading to a contradiction. Let q be a state in Q and let u be such that $\delta(u) = q$. The state q can be identified with the state $\delta_{min}(u)$. This identification is consistent and defines a one to one correspondence between Q and Q_{min}. □

Second, we give a minimization algorithm for finding the minimum state DFTA equivalent to a given reduced DFTA. We confuse an equivalence relation and the sequence of its equivalence classes.

```
Minimization Algorithm MIN
input: complete and reduced DFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \delta)$
begin
    Set $P$ to $\{Q_f, Q - Q_f\}$ /* P is the initial equivalence relation*/
    repeat
        $P' = P$
        /* Refine equivalence $P$ in $P'$ */
        $qP'q'$ if
            $qPq'$ and
            $\forall f \in \mathcal{F}_n \forall q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n \in Q$
            $\delta(f(q_1, \dots, q_{i-1}, q, q_{i+1}, \dots, q_n)) P \delta(f(q_1, \dots, q_{i-1}, q', q_{i+1}, \dots, q_n))$
    **until** $P' = P$
    Set $Q_{min}$ to the set of equivalence classes of $P$
    /* we denote by $[q]$ the equivalence class of state $q$ w.r.t. $P$ */
    Set $\delta_{min}$ to $\{f([q_1], \dots, [q_n]) \to [f(q_1, \dots, q_n)]\}$
    Set $Q_{min_f}$ to $\{[q] \mid q \in Q_f\}$
    **output:** DFTA $\mathcal{A}_{min} = (Q_{min}, \mathcal{F}, Q_{min_f}, \delta_{min})$
**end**
```

The DFTA constructed by the algorithm $\mathcal{MIN}$ is the minimum state DFTA for its tree language. Indeed, let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ the DFTA to which is applied the algorithm and let $L = L(\mathcal{A})$. Let $\mathcal{A}_{min}$ be the output of the algorithm. It is easy to show that the definition of $\mathcal{A}_{min}$ is consistent and that $L = L(\mathcal{A}_{min})$. Now, by contradiction, we can prove that $\mathcal{A}_{min}$ has no more states than the number of equivalence classes of $\equiv_L$.

1.6 Top Down Tree Automata

Tree automata that we have defined in the previous sections are also known as bottom-up tree automata because these automata start their computation at the leaves of trees. In this section we define top-down tree automata. Such an automaton starts its computation at the root in an initial state and then simultaneously works down the paths of the tree level by level. The tree automaton accepts a tree if a run built up in this fashion can be defined. It appears that top-down tree automata and bottom-up tree automata have the same expressive power. An important difference between bottom-up tree automata and top-down automata appears in the question of determinism since deterministic top-down tree automata are strictly less powerful than nondeterministic ones and therefore are strictly less powerful that bottom-up tree automata. Intuitively, it is due to the following: tree properties specified by deterministic top-down tree automata can depend only on path properties. We now make precise these remarks and first, let us formally define top-down tree automata.

A nondeterministic ***top-down*** finite Tree Automaton (top-down NFTA) over $\mathcal{F}$ is a tuple $\mathcal{A} = (Q, \mathcal{F}, I, \Delta)$ where Q is a set of states (states are unary symbols), $I \subseteq Q$ is a set of initial states, and Δ is a set of rewrite rules of the following type :

$$q(f(x_1, \dots, x_n)) \to f(q_1(x_1), \dots, q_n(x_n)),$$

where $n \geq 0$, $f \in \mathcal{F}_n$, $q, q_1, \dots, q_n \in Q$, $x_1, \dots, x_n \in \mathcal{X}$.

When $n = 0$, i.e. when the symbol is a constant symbol a, a transition rule of top-down NFTA is of the form $q(a) \to a$. A top-down automaton starts at the root and moves downward, associating along a run a state with each subterm inductively. We do not define formally define the move relation $\to_{\mathcal{A}}$ defined by a top-down NFTA because the definition is easily deduced from the corresponding definition for bottom-up NFTA. The tree language $L(\mathcal{A})$ recognized by $\mathcal{A}$ is the set of all ground terms t for which there is an initial state q in I such that

$$q(t) \xrightarrow[\mathcal{A}]{*} t.$$

The expressive power of bottom-up and top-down tree automata is the same. Indeed, we have the following Theorem:

Theorem 8 (The equivalence of top-down and bottom-up NFTA's). *The class of languages accepted by top-down NFTA's is exactly the class of recognizable tree languages.*

Proof. The proof is left to the reader. **Hint.** Reverse the arrows and exchange the sets of initial and final states. □

Top-down and bottom-up tree automata have the same expressive power because they define the same classes of tree languages. Nevertheless they do not have the same behavior from an algorithmic point of view because nondeterminism can not be reduced in the class of top-down tree automata.

Proposition 1 (Top-down NFTA's and top-down DFTA's). *A top-down finite Tree Automaton $(Q, \mathcal{F}, I, \Delta)$ is deterministic (top-down DFTA) if there is one initial state and no two rules with the same left-hand side. Top-down DFTA's are strictly less powerful than top-down NFTA's, i.e. there exists a recognizable tree language which is not accepted by a top-down DFTA.*

Proof. Let $\mathcal{F} = \{f(,), a, b\}$. And let us consider the recognizable tree language $T = \{f(a, b), f(b, a)\}$. Now let us suppose there exists a top-down DFTA that accepts T, the automaton should accept the term $f(a, a)$ leading to a contradiction. □

1.7 Decision problems and their complexity

In this section, we study some decision problems and their complexity. The size of an automaton will be the size of its representation. More formally:

Definition 1. *Let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ be an NFTA over $\mathcal{F}$. The size of a rule $f(q_1(x_1), \dots, q_n(x_n)) \to q(f(x_1, \dots, x_n))$ is $arity(f) + 1$. The size of $\mathcal{A}$ noted $\|A\|$, is defined by:*

$$\|\mathcal{A}\| = |Q| + \sum_{f(q_1(x_1),\dots,q_n(x_n)) \to q(f(x_1,\dots,x_n)) \in \Delta} (arity(f) + 2).$$

We will work in the frame of RAM machines, with uniform measure.

Membership

Instance A tree automaton and a ground term.

Answer "yes" if and only if the term is recognized by the automaton.

Clearly, the recognizable tree languages are recursive as a NFTA can be viewed as an acceptance algorithm.

Theorem 9. *Membership can be decided in linear time for DFTA, in polynomial time for NFTA.*

Proof. In the deterministic case, an algorithm for testing membership in $O(\|t\| + \|\mathcal{A}\|)$ is easy to obtain. For the nondeterministic case, the idea is similar as in the word case: the algorithm determinizes along the computation. The complexity of the algorithm will be in $O(\|t\| \times \|\mathcal{A}\|)$. □

Emptiness

Instance A tree automaton

Answer “yes” if and only if the recognized language is empty.

Theorem 10. *It can be decided in linear time whether the language accepted by a finite tree automaton is empty.*

Proof. The minimal height of accepted terms can be bounded by the number of states using Corollary 1; so, as membership is decidable, emptiness is decidable. Of course, this approach does not provide a practicable algorithm. To get an efficient algorithm, it suffices to notice that a NFTA accepts at least one tree if and only if there is an accessible final state: this algorithm can be viewed as a least fix-point computation.In other words, the language recognized by a **reduced** automaton is empty if and only if the set of final states is non empty. Reducing an automaton can be done in $O(|Q| \times \|\mathcal{A}\|)$ by the reduction algorithm given in Section 1.1. Actually, this algorithm can be improved by choosing an adequate data structure in order to get a linear algorithm (see Exercise 16). This linear least fixpoint computation holds in serveral frameworks. For example, it can be viewed as the satisfiability test of a set of propositional Horn formulae. The reduction is easy and linear: each state q can be associated with a propositional variable X_q and each rule $r : f(q_1, \ldots, q_n) \rightarrow q$ can be associated with a propositional Horn formula $F_r = X_q \vee \neg X_{q_1} \vee \cdots \vee \neg X_{q_n}$. It is straightforward that satisfiability of $\{F_r\} \cup \{\neg X_q / q \in Q_f\}$ is equivalent to emptiness of the language recognized by $(Q, \mathcal{F}, Q_f, \Delta)$. So, as satisfiability of a set of propositional Horn formulae can be decided in linear time, we get a linear algorithm for testing emptiness for NFTA. □

The emptiness problem is **P-complete** with respect to logspace reductions, even when restricted to deterministic tree automata. The proof can easily be done since the problem is very close to *the solvable path systems* problem which is known to be P-complete (see Exercise 17).

Intersection non-emptiness

Instance A finite sequence of tree automata.

Answer “yes” if and only if there is at least one term recognized by each automaton of the sequence.

Theorem 11. *The intersection problem for tree automata is EXPTIME-complete.*

Proof. By constructing the product automata for the n automata, and then testing non-emptiness, we get an algorithm in $O(\|\mathcal{A}_1\| \times \cdots \times \|\mathcal{A}_n\|)$. The proof of EXPTIME-hardness is based on simulation of an alternating linear space-bounded Turing machine. Roughly speaking, with such a machine and an input of length n can be associated polynomially n tree automata whose intersection corresponds to the set of accepting computations on the input. It is worth noting that the result holds for deterministic top down tree automata as well as for deterministic bottom-up ones. □

Finiteness

Instance A tree automaton

Answer "yes" if and only if the recognized language is finite.

Theorem 12. *Finiteness can be decided in polynomial time.*

Proof. Let us consider a NFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$. Deciding finiteness of $\mathcal{A}$ is direct by Corollary 1: it suffices to find an accepted term t s.t. $|Q| < \|t\| \leq 2 * |Q|$. A more efficient way to test finiteness is to check the existence of a loop: the language is infinite if and only if there is a loop on some useful state, i.e. there exist an accessible state q and contexts C and C' such that $C[q] \xrightarrow[\mathcal{A}]{*} q$ and $C'[q] \xrightarrow[\mathcal{A}]{*} q'$ for some final state q'. For a given q, deciding if there is a loop on q can be done in linear time. So, finiteness can be decided in quadratic time, more precisely in $O(|Q| \times \|\mathcal{A}\|)$. □

Emptiness of the complement

Instance A tree automaton.

Answer "yes" if and only if every term is accepted by the automaton

Deciding whether a deterministic tree automaton recognizes the set of all terms is polynomial for a fixed alphabet: we just have to check whether the automaton is complete (which can be done in $O(|\mathcal{F}| \times |Q|^{Arity(\mathcal{F})})$) and then it remains only to check that all accessible states are final. For nondeterministic automata, the following result proves in some sense that determinization with its exponential cost is unavoidable:

Theorem 13 (Seidl [Sei90]). *The problem whether a tree automaton accepts the set of all terms is EXPTIME-complete for nondeterministic tree automata.*

Proof. The proof of this theorem is once more based on simulation of linear space bounded alternating Turing machine: indeed, the complement of the accepting computations on an input w can be coded polynomially in a recognizable tree language. □

Equivalence

Instance Two tree automata

Answer "yes" if and only if the automata recognize the same language.

Theorem 14. *Equivalence is decidable for tree automata.*

Proof. Clearly, as the class of recognizable sets is effectively closed under complementation and intersection, and as emptiness is decidable, equivalence is decidable. For two deterministic complete automata $\mathcal{A}_1$ and $\mathcal{A}_2$, we get by these means an algorithm in $O(\|\mathcal{A}_1\| \times \|\mathcal{A}_2\|)$. (An other way is to compare the minimal automata). For nondeterministic ones, this approach leads to an exponential algorithm. □

As we have proved that deciding whether an automaton recognizes the set of all ground terms is EXPTIME-hard, we get immediately:

Corollary 3. *The inclusion problem and the equivalence problem for NFTA's are EXPTIME-complete.*

1.8 Exercises

Exercise 1. Let $\mathcal{F} = \{f(,), g(), a\}$. Define a top-down NFTA, a NFTA and a DFTA for the set $G(t)$ of ground instances of term $t = f(f(a, x), g(y))$. Is it possible to define a top-down DFTA for this language?

Exercise 2. Let $\mathcal{F} = \{f(,), g(), a\}$. Define a top-down NFTA, a NFTA and a DFTA for the set of terms which have a ground instance of term $t = f(a, g(y))$ as a subterm. Is it possible to define a top-down DFTA for this language?

Exercise 3. Let $\mathcal{F} = \{g(), a\}$. Is the set of ground terms whose height is even recognizable? Let $\mathcal{F} = \{f(,), g(), a\}$. Is the set of ground terms whose height is even recognizable?

Exercise 4. Let $\mathcal{F} = \{f(,), a\}$. Prove that the set $L = \{f(t, t) \mid t \in T(\mathcal{F})\}$ is not recognizable. Let $\mathcal{F}$ be any ranked alphabet which contains at least one constant symbol a and one binary symbol $f(,)$. Prove that the set $L = \{f(t, t) \mid t \in T(\mathcal{F})\}$ is not recognizable.

Exercise 5. Prove the equivalence between top-down NFTA and NFTA.

Exercise 6. Let $\mathcal{F} = \{f(,), g(), a\}$ and $\mathcal{F}' = \{f'(,), g(), a\}$. Let us consider the tree homomorphism h determined by $h_{\mathcal{F}}$ defined by: $h_{\mathcal{F}}(f) = f'(x_1, x_2)$, $h_{\mathcal{F}}(g) = f'(x_1, x_1)$, and $h_{\mathcal{F}}(a) = a$. Is $h(T(\mathcal{F}))$ recognizable? Let $L_1 = \{g^i(a) \mid i \geq 0\}$, then L_1 is a recognizable tree language, is $h(L_1)$ recognizable? Let L_2 be the recognizable tree language defined by $L_2 = L(\mathcal{A})$ where $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ is defined by: $Q = \{q_a, q_g, q_f\}$, $Q_f = \{q_f\}$, and Δ is the following set of transition rules:

$$\begin{array}{lrcllrcl}
\{ & a & \to & q_a & & g(q_a) & \to & q_g \\
 & f(q_a, q_a) & \to & q_f & & f(q_g, q_g) & \to & q_f \\
 & f(q_a, q_g) & \to & q_f & & f(q_g, q_a) & \to & q_f \\
 & f(q_a, q_f) & \to & q_f & & f(q_f, q_a) & \to & q_f \\
 & f(q_g, q_f) & \to & q_f & & f(q_f, q_g) & \to & q_f \\
 & f(q_f, q_f) & \to & q_f & \}. & & &
\end{array}$$

Is $h(L_2)$ recognizable?

Exercise 7. Let $\mathcal{F}_1 = \{or(,), and(,), not(), 0, 1, x\}$. A ground term over $\mathcal{F}$ can be viewed as a boolean formula over variable x. Define a DFTA which recognizes the set of satisfiable boolean formulae over x. Let $\mathcal{F}_n = \{or(,), and(,), not(), 0, 1, x_1, \ldots, x_n\}$. A ground term over $\mathcal{F}$ can be viewed as a boolean formula over variables $x_1, \ldots, x_n$. Define a DFTA which recognizes the set of satisfiable boolean formulae over $x_1, \ldots, x_n$.

Exercise 8. Let t be a linear term in $T(\mathcal{F}, \mathcal{X})$. Prove that the set $G(t)$ of ground instances of term t is recognizable. Let R be a finite set of linear terms in $T(\mathcal{F}, \mathcal{X})$. Prove that the set $G(R)$ of ground instances of set R is recognizable.

Exercise 9. Let R be a finite set of linear terms in $T(\mathcal{F}, \mathcal{X})$. We define the set $\mathit{Red}(R)$ of reducible terms for R to be the set of ground terms which have a ground instance of some term in R as a subterm. Prove that the set $\mathit{Red}(R)$ is recognizable.

Exercise 10. We consider the following two problems. First, given as instance a recognizable tree language L and a tree homomorphism h, is the set $h(L)$ recognizable? Second, given as instance a set R of terms in $T(\mathcal{F}, \mathcal{X})$, is the set $\mathit{Red}(R)$ recognizable? Prove that if the first problem is decidable, the second problem is easily shown decidable.

Exercise 11. Let R be a finite set of linear terms in $T(\mathcal{F}, \mathcal{X})$. A term t is inductively reducible for R if all the ground instances of term t are reducible for R. Prove that inductive reducibility of a linear term t for a set of linear terms R is decidable.

Exercise 12. Let $\mathcal{F} = \{f(,), a, b\}$.

1. Let us consider the set of ground terms L_1 defined by the following two conditions:
   - $f(a, b) \in L_1$,
   - $t \in L_1 \Rightarrow f(a, f(t, b)) \in L_1$.

   Prove that the set L_1 is recognizable.
2. Prove that the set $L_2 = \{t \in T(\mathcal{F}) \mid |t|_a = |t|_b\}$ is not recognizable where $|t|_a$ (respectively $|t|_b$) denotes the number of a (respectively the number of b) in t.
3. Let L be a recognizable tree language over $\mathcal{F}$. Let us suppose that f is a commutative symbol. Let $C(L)$ be the congruence closure of set L for the set of equations $C = \{f(x, y) = f(y, x)\}$. Prove that $C(L)$ is recognizable.
4. Let L be a recognizable tree language over $\mathcal{F}$. Let us suppose that f is a commutative and associative symbol. Let $AC(L)$ be the congruence closure of set L for the set of equations $AC = \{f(x, y) = f(y, x); f(x, f(y, z)) = f(f(x, y), z)\}$. Prove that in general $AC(L)$ is not recognizable.
5. Let L be a recognizable tree language over $\mathcal{F}$. Let us suppose that f is an associative symbol. Let $A(L)$ be the congruence closure of set L for the set of equations $A = \{f(x, f(y, z)) = f(f(x, y), z)\}$. Prove that in general $A(L)$ is not recognizable.

Exercise 13. Consider the *complement problem*:

- **Instance** A term $t \in T(\mathcal{F}, \mathcal{X})$ and terms $t_1, \ldots, t_n$,
- **Question** There is a ground instance of t which is not an instance of any t_i.

Prove that the complement problem is decidable whenever term t and all terms t_i are linear. Extend the proof to handle the case where t is a term (not necessarily linear).

Exercise 14. Let $\mathcal{F}$ be a ranked alphabet and suppose that $\mathcal{F}$ contains some symbols which are commutative and associative. The set of ground AC-instances of a term t is the AC-congruence closure of set $G(t)$. Prove that the set of ground AC-instances of a linear term is recognizable. The reader should note that the set of ground AC-instances of a set of linear terms is not recognizable (see Exercice refexocloturecetacderec).

Prove that the *AC-complement problem* is decidable where the AC-complement problem is defined by:

- **Instance** A linear term $t \in T(\mathcal{F}, \mathcal{X})$ and linear terms $t_1, \ldots, t_n$,
- **Question** There is a ground AC-instance of t which is not an AC-instance of any t_i.

Exercise 15. Let $\mathcal{F}$ be a ranked alphabet and $\mathcal{X}$ be a countable set of variables. Let S be a rewrite system on $T(\mathcal{F}, \mathcal{X})$ (the reader is referred to [DJ90]) and L be a set of ground terms. We denote by $S^*(L)$ the set of reductions of terms in L by S and by $S(L)$ the set of ground S-normal forms of set L. Formally,

$$S^*(L) = \{t \in T(\mathcal{F}) \mid \exists u \in L \; u \xrightarrow{*} t\},$$

$$S(L) = \{t \in T(\mathcal{F}) \mid t \in IRR(S) \text{ and } \exists u \in L \; u \xrightarrow{*} t\} = IRR(S) \cap S^*(L).$$

We consider the two following decision problems:

(1rst order reachability)

- **Instance** A rewrite system S, two ground terms u and v,
- **Question** $v \in S^*(\{u\})$.

(2nd order reachability)

- **Instance** A rewrite system S, two recognizable tree languages L and L',
- **Question** $S^*(L) \subseteq L'$.

1. Let us suppose that rewrite system S satisfies:

   (PreservRec) If L is recognizable, then $S^*(L)$ is recognizable.

   What can be said about the two reachability decision problems? Give a sufficient condition on rewrite system S satisfying (PreservRec) such that S satisfies (NormalFormRec) where (NormalFormRec) is defined by:

   (NormalFormRec) If L is recognizable, then $S(L)$ is recognizable.

2. Let $\mathcal{F} = \{f(,), g(), h(), a\}$. Let $L = \{f(t_1, t_2) \mid t_1, t_2 \in T(\{g(), h(), a\}\}$, and S is the following set of rewrite rules:

   $$\left\{ \begin{array}{rcl@{\qquad}rcl} f(g(x), h(y)) & \rightarrow & f(x, y) & f(h(x), g(y)) & \rightarrow & f(x, y) \\ g(h(x)) & \rightarrow & x & h(g(x)) & \rightarrow & x \\ f(a, x) & \rightarrow & x & f(x, a) & \rightarrow & x \end{array} \right\}$$

   Are the sets L, $S^*(L)$, and $S(L)$ recognizable?

3. Let $\mathcal{F} = \{f(,), g(), h(), a\}$. Let $L = \{g(h^n(a)) \mid n \geq 0\}$, and S is the following set of rewrite rules:

   $$\{ \quad g(x) \rightarrow f(x, x) \quad \}$$

   Are the sets L, $S^*(L)$, and $S(L)$ recognizable?

4. Let us suppose now that rewrite system S is linear and monadic, i.e. all rewrite rules are of one of the following three types:

   $$\begin{array}{lccll} (1) & l & \rightarrow & a & , a \in \mathcal{F}_0 \\ (2) & l & \rightarrow & x & , x \in \mathcal{V}ar(l) \\ (3) & l & \rightarrow & f(x_1, \ldots, x_p) & , x_1, \ldots, x_p \in \mathcal{V}ar(l), f \in \mathcal{F}_p \end{array}$$

   where l is a linear term (no variable occurs more than once in t) whose height is greater than 1. Prove that a linear and monadic rewrite system satisfies (PreservRec). Prove that (PreservRec) is false if the right-hand side of rules of type (3) may be non linear.

Exercise 16. Design a linear-time algorithm for testing emptiness of the language recognized by a tree automaton:

Instance A tree automaton

Answer "yes" if and only if the language recognized is empty.

Hint: Choose a suitable data structure for the automaton. For example, a state could be associated with the list of the "adresses" of the rules whose left-hand side contain it (eventually, a rule can be repeated); each rule could be just represented by a counter initialized at the arity of the corresponding symbol and by the state of the right-hand side. Activating a state will decrement the counters of the corresponding rules. When the counter of a rule becomes null, the rule can be applied: the right-hand side state can be activated.

Exercise 17.
The Solvable Path Problem is the following:

Instance a finite set X and three sets $R \subset X \times X \times X, X_s \subset X$ and $X_t \subset X$.

Answer "yes" if and only if $X_t \cap A$ is non empty, where A is the least subset A of X such that $X_s \subset A$ and if $y, z \in A$ and $(x, y, z) \in R$, then $x \in A$.

Prove that this $P - complete$ problem is log-space reducible to the emptiness problem for tree automata.

Exercise 18. A *flat automaton* is a tree automaton which has the following property: there is an ordering $\geq$ on the states and a particular state $q_\top$ such that the transition rules have one of the following forms:

1. $f(q_\top, \ldots, q_\top) \rightarrow q_\top$
2. $f(q_1, \ldots, q_n) \rightarrow q$ with $q > q_i$ for every i
3. $f(q_\top, \ldots, q_\top, q, q_\top, \ldots, q_\top) \rightarrow q$

Moreover, we assume that all terms are accepted in the state $q_\top$. (The automaton is called *flat* because there are no "nested loop").

Prove that the intersection of two flat automata is a finite union of automata whose size is linear in the sum of the original automata. (This contrasts with the construction of Theorem 5 in which the intersection automaton's size is the product of the sizes of its components).

Deduce from the above result that the intersection non-emptiness problem for flat automata is in NP (compare with Theorem 11).

1.9 Bibliographic Notes

Tree automata were introduced by Doner [Don65, Don70] and Thatcher and Wright [TW65, TW68]. Their goal was to prove the decidability of the weak second order theory of multiple successors. The original definitions are based on the algebraic approach and involve heavy use of universal algebra and/or category theory.

Many of the basic results presented in this chapter are the straightforward generalization of the corresponding results for finite automata. It is difficult to attribute a particular result to any one paper. Thus, we only give a list of some important contributions consisting of the above mentioned papers of Doner, Thatcher and Wright and also Eilenberg and Wright [EW67], Thatcher [Tha70], Brainerd [Bra68, Bra69], Arbib and Give'on [AG68]. All the results of this chapter and a more complete and detailed list of references can be found in the textbook of Gécseg and Steinby [GS84] and also in their recent survey [GS96]. For an overview of the notion of recognizability in general algebraic structures see Courcelle [Cou89] and the fundamental paper of Mezei and Wright [MW67].

In Nivat and Podelski [NP89] and [Pod92], the theory of recognizable tree languages is reduced to the theory of recognizable sets in an infinitely generated free monoid.

The results of Sections 1.1, 1.2, and 1.3 were noted in many of the papers mentioned above, but, in this textbook, we present these results in the style of the undergraduate textbook on finite automata by Hopcroft and Ullman [HU79]. Tree homomorphisms were defined as a special case of tree transducers, see Thatcher [Tha73]. The reader is referred to the bibliographic notes in Chapter 6 of the present textbook for detailed references. The reader should note that our proof of preservation of recognizability by tree homomorphisms and inverse tree homomorphisms is a direct construction using FTA. A more classical proof can be found in [GS84] and uses regular tree grammars (see Chapter 2).

Minimal tree recognizers and Nerode's congruence appear in Brainerd [Bra68, Bra69], Arbib and Give'on [AG68], and Eilenberg and Wright [EW67]. The proof we presented here is by Kozen [Koz92] (see also Fülöp and Vágvölgyi [FV89]). Top-down tree automata were first defined by Rabin [Rab69]. The reader is referred to [GS84] and [GS96] for more references and for the study of some subclasses of recognizable tree languages such as the tree languages recognized by deterministic top-down tree automata.

Some results of Sections 1.7 are "folklore" results. Many interesting results concerning complexity and tree automata can be found in Seidl [Sei89], [Sei90]. The EXPTIME-hardness of the problem of intersection non-emptiness is often used; this problem is close to problems of type inference and an idea of the proof can be found in [FSVY91]. A proof for deterministic top-down automata can be found in [Sei94b]. A detailed proof in the deterministic bottom-up case as well as some other results about complexity and tree automata can be found in [Vea97a], [Vea97b].

Recently, applications of tree automata theory to automated deduction and to the theory of rewriting systems were studied. Numerous exercises of the present section introduce these applications. These applications are studied in more details in Section 3.4. Most of the recent results about tree automata and rewrite systems are collected in Gilleron and Tison [GT95]. Let S be a term rewrite system (see for example Dershowitz and Jouannaud [DJ90] for a survey on rewrite systems), if S is left-linear the set $IRR(S)$ of irreducible ground terms w.r.t. S is a recognizable tree language. This result first appears in Gallier and Book [GB85] and is the subject of Exercise 9. However not every recognizable tree language is the set of irreducible terms w.r.t. a rewrite system S (see Fülöp and Vágvölgyi [FV88]). It was proved that the problem whether, given a rewrite system S as instance, the set of irreducible terms is recognizable is decidable (Kucherov [Kuc91]). The problem of preservation of regularity by tree homomorphisms is not known decidable. Exercise 10 shows that this problem is stronger than the previous one.

The notion of inductive reducibility (or ground reducibility) was introduced in automated deduction. A term t is S-inductively (or S-ground) reducible for S if all the ground instances of term t are reducible for S. Inductive reducibility is decidable for linear term t and left-linear rewrite system S. This is Exercise 11, see also Section 3.4.2. Inductive reducibility is decidable for finite S (see Plaisted [Pla85]). Complement problems are also introduced in automated deduction. They are the subject of Exercises 13 and 14. The complement problem for linear terms was proved decidable by Lassez and Marriott [LM87] and the AC-complement problem by Lugiez and Moysset [LM94].

The reachability problem is defined in Exercise 15. It is well known that this problem is undecidable in general. It is decidable for rewrite systems preserving recognizability, i.e. such that for every recognizable tree language L, the set of reductions of terms in L by S is recognizable. This is true for linear and monadic rewrite systems (right-hand sides have depth less than 1). This result was obtained by K. Salomaa [Sal88] and is the matter of Exercise 15. This is true also for linear and semi-monadic (variables in the right-hand sides have depth at most 1) rewrite systems, Coquidé et al. [CDGV94]. Other interesting results can be found in [Jac96].

Chapter 2

Regular grammars and regular expressions

2.1 Tree Grammar

In the previous chapter, we have studied regular tree languages from the acceptor point of view, using tree automata. In this chapter we study regular languages from the generation point of view, using tree grammars. Again, we shall find that many properties and concepts on word languages smoothly generalize to tree languages and that algebraic characterization of regular languages do exist for tree languages. Actually, this is not surprising since tree languages can be seen as word languages on an infinite alphabet of contexts with one hole. We shall show also that the set of derivations of a context-free language is a regular tree language.

2.1.1 Definitions

When we write programs, we often have to know how to produce the elements of the data structures that we use. For instance, a definition of the lists of integers in a functional language like ML is similar to the following definition:

$$\begin{array}{l} Nat = 0 \mid s(Nat) \\ List = nil \mid cons(Nat, List) \end{array}$$

This definition is nothing but a tree grammar in disguise, more precisely the set of lists of integers is the tree language generated by the grammar with axiom $List$, non-terminal symbols $List, Nat$, terminal symbols $0, s, nil, cons$ and rules

$$\begin{array}{lcl} Nat & \rightarrow & 0 \\ Nat & \rightarrow & s(Nat) \\ List & \rightarrow & nil \\ List & \rightarrow & cons(Nat, List) \end{array}$$

Tree grammars are similar to word grammars except that basic objects are trees, therefore terminals and non-terminals may have an arity greater than 0. More precisely, a ***tree grammar*** $G = (A, N, \mathcal{F}, R)$ is composed of an ***axiom*** A, a set N of ***non-terminal*** symbols with $A \in N$, a set $\mathcal{F}$ of ***terminal*** symbols, a set R of ***production rules*** of the form $\alpha \rightarrow \beta$ where α, β are trees of $T(\mathcal{F} \cup N \cup \mathcal{X})$ where $\mathcal{X}$ is a set of dummy variables and α contains at least one non-terminal. Moreover we require that $\mathcal{F} \cap N = \emptyset$, that each element of $N \cup \mathcal{F}$ has a fixed arity and that the arity of the axiom A is 0. In this chapter, we shall concentrate on ***regular tree grammars*** where a regular tree grammar $G = (A, N, \mathcal{F}, R)$ is a tree grammar such that all non-terminal symbols have arity 0 and production rules have the form $X \rightarrow \beta$, with X a non-terminal of N and β a tree of $T(\mathcal{F} \cup N)$.

Example 15. The grammar G with axiom $List$, non-terminals $List, Nat$ terminals $0, nil, s(), cons(,)$, rules

$$\begin{array}{l} List \rightarrow nil \\ List \rightarrow cons(Nat, List) \\ Nat \rightarrow 0 \\ Nat \rightarrow s(Nat) \end{array}$$

is a regular tree grammar.

A tree grammar is used to build terms from the axiom, using the corresponding ***derivation relation***. Basically the idea is to replace a non-terminal X by the right-hand side α of a rule $X \rightarrow \alpha$. More precisely, given a regular tree grammar $G = (A, N, \mathcal{F}, R)$, the derivation relation $\rightarrow$ associated to G is a relation on pairs of terms of $T(\mathcal{F} \cup N)$ such that $s \rightarrow t$ if and only if there are a rule $X \rightarrow \alpha \in R$, a context C such that $s = C[X]$ and $t = C[\alpha]$. The ***language generated*** by G, denoted by $L(G)$, is the set of terms of $T(\mathcal{F})$ which can be reached by successive derivations starting from the axiom, i.e. $L(G) = \{s \in T_{\mathcal{F}} \mid A \xrightarrow{+} s\}$ with $\xrightarrow{+}$ the transitive closure of $\rightarrow$.

Example 16. Let G be the grammar of the previous example, then a derivation of $cons(s(0), nil)$ from $List$ is

$$List \rightarrow_G cons(Nat, List) \rightarrow_G cons(s(Nat), List) \rightarrow_G cons(s(Nat), nil) \rightarrow_G cons(s(0), nil)$$

and the language generated by G is the set of lists of non-negative integers.

From the example, we can see that trees are generated top-down by replacing a leaf by some other term. When X is a non-terminal of a regular tree grammar G, we denote by $L_G(X)$ the language generated by the grammar G' identical to G but with X as axiom. When there is no ambiguity on the grammar referred to, we drop the subscript G. We say that two grammars G and G' are ***equivalent*** when they generate the same language. Grammars can contain useless rules or non-terminals and we want to get rid of these while preserving the generated language. A non-terminal is ***reachable*** if there is a derivation from the axiom containing this non-terminal. A non-terminal X is ***productive*** if $L_G(X)$ is non-empty. A regular tree grammar is ***reduced*** if and only if all its non-terminals are reachable and productive. We have the following result:

Proposition 2. *A regular tree grammar is equivalent to a normalized regular tree grammar.*

Proof. Given a grammar $G = (A, N, \mathcal{F}, R)$, we can compute the set of reachable terminals and the set of productive terminals using the sequences $(Reach)_n$ and $(Prod)_n$ which are defined in the following way.

$$\begin{aligned} Prod_0 &= \emptyset \\ Prod_n &= \ Prod_{n-1} \\ &\quad \cup \\ &\quad \{X \in N \mid \exists (X \to \alpha) \in R \text{ and each non-terminal of } \alpha \text{ is in } Prod_{n-1}\} \end{aligned}$$

and

$$\begin{aligned} Reach_0 &= \{A\} \\ Reach_n &= \ Reach_{n-1} \\ &\quad \cup \\ &\quad \{X \in N \mid \exists (X' \to \alpha) \in R \text{ such that } X' \in Reach_{n-1} \text{ and } X \text{ occurs in } \alpha\} \end{aligned}$$

For each sequence, there is an index such that all elements of the sequence with greater index are identical and this element is the set of productive (resp. reachable) non-terminals of G. Each regular tree grammar is equivalent to a reduced tree grammar which is computed by the following cleaning algorithm.

Computation of an equivalent reduced grammar
input: a regular tree grammar $G = (A, N, \mathcal{F}, R)$.

1. Compute the set of productive literals N_{Prod} using the sequence $(Prod)_n$ and let $G' = (A, N', \mathcal{F}, R)$ where $N' = N \cap N_{Prod}$ and R' is the subset of R involving rules containing only productive non-terminals.

2. Compute the set N_{Reach} of reachable terminals N_{Reach} using the sequence $(Reach)_n$ and let $G'' = (A, N' \cap N_{Reach}, \mathcal{F}, R'')$ where R'' is the subset of R' involving rules containing only reachable terminals.

output: G''

The equivalence of G, G' and G'' is straightforward. Moreover each non-terminal X of G'' must appear in a derivation $A \xrightarrow{*} C[X] \xrightarrow{*} C[s]$ which proves that G'' is reduced. The reader should notice that to exchange the two steps of the computation may result in a grammar which is not reduced (see Exercise 21). □

Actually, we shall use even simpler grammars, i.e. ***normalized*** regular tree grammar, where the production rules have the form $X \to f(X_1, \ldots, X_n)$ or $X \to a$ where f, a are symbols of $\mathcal{F}$ and $X, X_1, \ldots, X_n$ are non-terminals. The following result shows that this is not a restriction.

Proposition 3. *A regular tree grammar is equivalent to a normalized regular tree grammar.*

Proof. Replace a rule $X \to f(s_1, \ldots, s_n)$ by $X \to f(X_1, \ldots, X_n)$ with $X_i = s_i$ if $s_i \in N$ otherwise X_i is a new non-terminal. In the last case add the rule $X_i \to s_i$. Iterate this process until one gets a (necessarily equivalent) grammar with rules of the form $X \to f(X_1, \ldots, X_n)$ or $X \to a$ or $X_1 \to X_2$. The last rules are replaced by the rules $X_1 \to \alpha$ for all $\alpha \notin N$ such that $X_1 \xrightarrow{+} X_i$ and $X_i \to \alpha \in R$ (these $X_i^\prime s$ are easily computed using a transitive closure algorithm). □

From now on, we assume that all grammars are normalized, unless this is stated otherwise explicitly.

2.1.2 Regular tree grammar and recognizable tree languages

Given some normalized regular tree grammar $G = (A, N, \mathcal{F}, R_G)$, we show how to build a top-down tree automaton which recognizes $L(G)$. We define $\mathcal{A} = (Q, \mathcal{F}, I, \Delta)$ by

- $Q = \{q_X \mid X \in N\}$
- $I = \{q_A\}$
- $q_X(f(x_1, \dots, x_n)) \rightarrow f(q_{X_1}(x_1), \dots, q_{X_n}(x_n)) \in \Delta$ if and only if $X \rightarrow f(X_1, \dots, X_n) \in R_G$.

A standard proof by induction on derivation length yields $L(G) = L(\mathcal{A})$. Therefore we have proved that the languages generated by regular tree grammar are recognizable languages.

The next question to ask is whether recognizable tree languages can be generated by regular tree grammars. If L is a regular tree language, there exists a top-down tree automata $\mathcal{A} = (Q, \mathcal{F}, I, \Delta)$ such that $L = L(\mathcal{A})$. We define $G = (A, N, \mathcal{F}, R_G)$ with A a new symbol, $N = \{X_q \mid q \in Q\}$, $R_G = \{X_q \rightarrow f(X_{q_1}, \dots, X_{q_n}) \mid q(f(x_1, \dots, x_n)) \rightarrow f(q_1(x_1), \dots, q_n(x_n)) \in R\} \cup \{A \rightarrow X_I \mid X_I \in I\}$. A standard proof by induction on derivation length yields $L(G) = L(\mathcal{A})$.

Combining these two properties, we get the equivalence between recognizable languages and the languages generated by regular tree grammars.

Theorem 15. *A tree language is recognizable if and only if it is generated by a regular tree grammar.*

2.2 Regular expressions. Kleene's theorem for tree languages.

Going back to our example of lists of nonnegative integers, we can write the sets defined by the non-terminals Nat and $List$ as follows.

$$\begin{aligned} Nat &= \{0, s(0), s(s(0)), \dots\} \\ List &= \{nil, cons(\_, nil), cons(\_, cons(\_, nil), \dots\} \end{aligned}$$

where $\_$ stands for any element of Nat. There is some regularity in each set which reminds of the regularity obtained with regular word expressions constructed with the union, concatenation and iteration operators. Therefore we can try to use the same idea to denote the sets Nat and $List$. However, since we are dealing with trees and not words, we must put some information to indicate where concatenation and iteration must take place. This is done by using a new symbol which behaves as a constant. Moreover, since we have two independent iterations, the first one for Nat and the second one for $List$, we shall use two different new symbols $\Box_1$ and $\Box_2$ and a natural extension of regular word expression leads us to denote the sets Nat and $List$ as follows.

$$Nat = s(\square_1)^{*,\square_1} \;.\!_{\square_1}\; 0$$
$$List = nil + cons(\;(s(\square_1)^{*,\square_1} \;.\!_{\square_1}\; 0)\;, \square_2)^{*,\square_2} \;.\!_{\square_2}\; nil$$

We are going to show that this is a general phenomenon and that we can define a notion of regular expressions for trees and that Kleene's theorem for words can be generalized to trees. Like in the example, we must introduce a particular set of constants $\mathcal{K}$ which are used to indicate the positions where concatenation and iteration take place in trees. This explains why the syntax of regular tree expressions is more cumbersome than the syntax of word regular expressions. These new constants are usually denoted by $\square_1, \square_2, \ldots$. Therefore, in this section, we consider trees constructed on $\mathcal{F} \cup \mathcal{K}$ where $\mathcal{K}$ is a distinguished finite set of symbols of arity 0 disjoint from $\mathcal{F}$.

2.2.1 Substitution and iteration

First, we have to generalize the notion of substitution to languages, replacing some $\square_i$ by a tree of some language L_i. The main difference with term substitution is that different occurrences of the same constant $\square_i$ can be replaced by different terms of L_i. Given a tree t of $T(\mathcal{F} \cup \mathcal{K})$, $\square_1, \ldots, \square_n$ symbols of $\mathcal{K}$ and $L_1, \ldots, L_n$ languages of $T(\mathcal{F} \cup \mathcal{K})$, the ***tree substitution*** (substitution for short) of $\square_1, \ldots, \square_n$ by $L_1, \ldots, L_n$ in t, denoted by $t\{\square_1 \leftarrow L_1, \ldots, \square_n \leftarrow L_n\}$, is the tree language defined by the following identities.

- $\square_i\{\square_1 \leftarrow L_1, \ldots, \square_n \leftarrow L_n\} = L_i$ for $i = 1, \ldots, n$,
- $a\{\square_1 \leftarrow L_1, \ldots, \square_n \leftarrow L_n\} = \{a\}$ for all $a \in \mathcal{F} \cup \mathcal{K}$ such that arity of a is 0 and $a \neq \square_1, \ldots, a \neq \square_n$,
- $f(s_1, \ldots, s_n)\{\square_1 \leftarrow L_1, \ldots, \square_n \leftarrow L_n\} = \{f(t_1, \ldots, t_n) \mid t_i \in s_i\{\square_1 \leftarrow L_1, \ldots, \square_n \leftarrow L_n\}\}$

Example 17. Let $\mathcal{F} = \{0, nil, s(), cons(,)\}$ and $\mathcal{K} = \{\square_1, \square_2\}$, let

$$t = cons(\square_1, cons(\square_1, \square_2))$$

and let

$$L_1 = \{0, s(0)\}$$

then

$$\begin{array}{rcl} t\{\square_1 \leftarrow L\} & = & \{cons(0, cons(0, \square_2)), \\ & & cons(0, cons(s(0), \square_2)), \\ & & cons(s(0), cons(0, \square_2)), \\ & & cons(s(0), cons(s(0), \square_2))\} \end{array}$$

Symbols of $\mathcal{K}$ are mainly used to distinguish places where the substitution must take place, and they are usually not relevant. For instance, if t is a tree on the alphabet $\mathcal{F} \cup \{\square\}$ and L be a language of trees on the alphabet $\mathcal{F}$, then the trees of $t\{\square \leftarrow L\}$ don't contain the symbol $\square$.

The substitution operation generalizes to languages in a straightforward way. When $L, L_1, \ldots, L_n$ are languages of $T(\mathcal{F} \cup \mathcal{K})$ and $\square_1, \ldots, \square_n$ are elements of $\mathcal{K}$, we define $L\{\Box_1 \leftarrow L_1, \ldots, \Box_n \leftarrow L_n\}$ to be the set $\bigcup_{t \in L}\{\ t\{\Box_1 \leftarrow L_1, \ldots, \Box_n \leftarrow L_n\}\}$.

Now, we can define the concatenation operation for tree languages. Given L and M two languages of $T_{\mathcal{F} \cup \mathcal{K}}$, and $\Box$ be a element of $\mathcal{K}$, the ***concatenation*** of M to L through $\Box$, denoted by $L\ ._{\Box}\ M$, is the set of trees obtained by substituting the occurrence of $\Box$ in trees of L by trees of M, i.e. $L\ ._{\Box}\ M = \bigcup_{t \in L}\{t\{\Box \leftarrow M\}\}$.

To define the closure of a language, we must define the sequence of successive iterations. Given L a language of $T(\mathcal{F} \cup \mathcal{K})$ and $\Box$ an element of $\mathcal{K}$, the sequence $L^{n,\Box}$ is defined by the equalities.

- $L^{0,\ \Box} = \{\Box\}$
- $L^{n+1,\ \Box} = L^{n,\ \Box} \cup L^{n,\ \Box}\ ._{\Box}\ L$

The ***closure*** $L^{*,\Box}$ of L is the union of all $L^{n,\ \Box}$ for nonnegative n, i.e., $L^{*,\Box} = \cup_n L^{n,\Box}$. From the definition, one gets that $\{\Box\} \subseteq L^{*,\Box}$ for any L.

Example 18. Let $\mathcal{F} = \{0, nil, s(), cons(,)\}$, let $L = \{0, cons(0, \Box)\}$ and $M = \{nil, cons(s(0), \Box)\}$, then

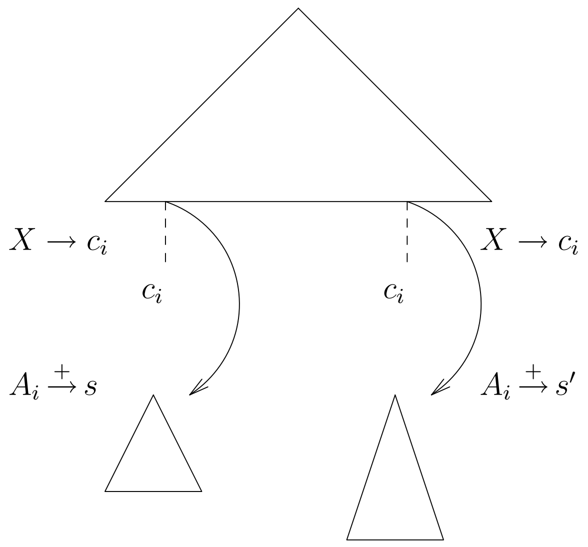
$$\begin{array}{rcl} L\ ._{\Box}\ M & = & \{0, cons(0, nil), cons(0, cons(s(0), \Box))\} \\ L^{*,\Box} & = & \{\Box\} \cup \\ & & \{0, cons(0, \Box)\} \cup \\ & & \{0, cons(0, nil), cons(0, cons(s(0), \Box))\} \cup \ldots \end{array}$$

We prove now that the substitution and concatenation operations yield regular languages when they are applied to regular languages.

Proposition 4. *Let L be a regular tree language on $\mathcal{F} \cup \mathcal{K}$, let $L_1, \ldots, L_n$ be regular tree languages on $\mathcal{F} \cup \mathcal{K}$, let $\Box_1, \ldots, \Box_n \in \mathcal{K}$, then $L\{\Box_1 \leftarrow L_1, \ldots, \Box_n \leftarrow L_n\}$ is a regular tree language.*

Proof. Since L is regular, there exists some normalized regular tree grammar $G = (A, N, \mathcal{F} \cup \mathcal{K}, R)$ such that $L = L(G)$, and for each $i = 1, \ldots, n$ there exists a normalized grammar $G_i = (A_i, N_i, \mathcal{F} \cup \mathcal{K}, R_i)$ such that $L_i = L(G_i)$. We can assume that the sets of non-terminals are pairwise disjoint. The idea of the proof is to construct a grammar G' which starts by generating trees like G but replaces the generation of a symbol $\Box_i$ by the generation of a tree of L_i via a branching towards the axiom of G_i. More precisely, we show that $L\{\Box_1 \leftarrow L_1, \ldots, \Box_n \leftarrow L_n\} = L(G')$ where $G' = (A, N', \mathcal{F} \cup \mathcal{K}, R')$ such that

- $N' = N \cup N_1 \cup \ldots \cup N_n$,
- R' contains the rules of R_i and the rules of R but the rules $X \rightarrow \Box_i$ which are replaced by the rules $X \rightarrow A_i$, where A_i is the axiom of L_i.



A straightforward induction on the height of trees proves that G' generates each tree of $L\{\square_1 \leftarrow L_1, \dots, \square_n \leftarrow L_n\}$.

The converse is to prove that $L(G') \subseteq L\{\square_1 \leftarrow L_1, \dots, \square_n \leftarrow L_n\}$. This is achieved by proving the following property by induction on the derivation length.

$$\begin{gathered} X \xrightarrow{+} s' \text{ where } s' \in T(\mathcal{F} \cup \mathcal{K}) \text{ using the rules of } G' \\ \text{if and only if} \\ \text{there is some } s \text{ such that } X \xrightarrow{+} s \text{ using the rules of } G \text{ and} \\ s' \in s\{\square_1 \leftarrow L_1, \dots, \square_n \leftarrow L_n\}. \end{gathered}$$

- base case: $X \rightarrow s$ in one step. Therefore this derivation is a derivation of the grammar G and no $\square_i$ occurs in s, yielding $s \in L\{\square_1 \leftarrow L_1, \dots, \square_n \leftarrow L_n\}$

- induction step: we assume that the property is true for any terminal and derivation of length less than n. Let X be such that $X \rightarrow s'$ in n steps. This derivation can be decomposed as $X \rightarrow s_1 \xrightarrow{+} s'$. We distinguish several cases depending on the rule used in the derivation $X \rightarrow s_1$.

  - the rule is $X \rightarrow f(X_1, \dots, X_m)$, therefore $s' = f(t_1, \dots, t_m)$ and $t_i \in L(X_i)\{\square_1 \leftarrow L_1, \dots, \square_n \leftarrow L_n\}$, therefore $s' \in L(X)\{\square_1 \leftarrow L_1, \dots, \square_n \leftarrow L_n\}$,
  - the rule is $X \rightarrow A_i$, therefore $X \rightarrow \square_i \in R$ and $s' \in L_i$ and $s' \in L(X)\{\square_1 \leftarrow L_1, \dots, \square_n \leftarrow L_n\}$.
  - the rule $X \rightarrow a$ with $a \in \mathcal{F}$, a of arity 0, $a \neq \square_1, \dots, a \neq \square_n$ are not considered since no further derivation can be done.

□

The following proposition states that regular languages are stable also under closure.

Proposition 5. *Let L be a regular tree language of $T(\mathcal{F} \cup \mathcal{K})$, let $\Box \in \mathcal{K}$, then $L^{*,\Box}$ is a regular tree language of $T(\mathcal{F} \cup \mathcal{K})$.*

Proof. There exists a normalized regular grammar $G = (A, N, \mathcal{F} \cup \mathcal{K}, R)$ such that $L = L(G)$ and we obtain from G a grammar $G' = (A', N \cup \{A'\}, \mathcal{F} \cup \mathcal{K}, R')$ for $L^{*,\Box}$ by replacing rules leading to $\Box$ such as $X \to \Box$ by rules $X \to A$ leading to the axiom. Moreover we add the rule $A' \to \Box$ to generate $\{\Box\} = L^{0,\Box}$ and the rule $A' \to A$ to generate $L^{i,\Box}$ for $i > 0$. By construction G' generates the elements of $L^{*,\Box}$.

Conversely a proof by induction on the length on the derivation proves that $L(G') \subseteq L^{*,\Box}$. $\Box$

2.2.2 Regular expressions and regular tree languages.

Now, we can define regular tree expression in the flavor of regular word expression using the $+, ._\Box, ^{*,\Box}$ operators.

Definition 2. *The set Regexp$(\mathcal{F}, \mathcal{K})$ of* ***regular tree expressions*** *on $\mathcal{F}$ and $\mathcal{K}$ is the smallest set such that:*

- *the empty set $\emptyset$ is in Regexp$(\mathcal{F}, \mathcal{K})$*
- *if $a \in \mathcal{F}_0 \cup \mathcal{K}$ is a constant, then $a \in$ Regexp$(\mathcal{F}, \mathcal{K})$,*
- *if $f \in \mathcal{F}_n$ has arity $n > 0$ and $E_1, \ldots, E_n$ are regular expressions of Regexp$(\mathcal{F}, \mathcal{K})$ then $f(E_1, \ldots, E_n)$ is a regular expression of Regexp$(\mathcal{F}, \mathcal{K})$,*
- *if E_1, E_2 are regular expressions of Regexp$(\mathcal{F}, \mathcal{K})$ then $(E_1 + E_2)$ is a regular expression of Regexp$(\mathcal{F}, \mathcal{K})$,*
- *if E_1, E_2 are regular expressions of Regexp$(\mathcal{F}, \mathcal{K})$ and $\Box$ is an element of $\mathcal{K}$ then $E_1 ._\Box E_2$ is a regular expression of Regexp$(\mathcal{F}, \mathcal{K})$,*
- *if E is a regular expression of Regexp$(\mathcal{F}, \mathcal{K})$ and $\Box$ is an element of $\mathcal{K}$ then $E^{*,\Box}$ is a regular expression of Regexp$(\mathcal{F}, \mathcal{K})$.*

Each regular expression E represents a set of terms of $T(\mathcal{F} \cup \mathcal{K})$ which we denote $[\![E]\!]$ and which is formally defined by the following equalities.

- $[\![\emptyset]\!] = \emptyset$,
- $[\![a]\!] = \{a\}$ for $a \in \mathcal{F}_0 \cup \mathcal{K}$,
- $[\![f(E_1, \ldots, E_n)]\!] = \{f(s_1, \ldots, s_n) \mid s_1 \in [\![E_1]\!], \ldots, s_n \in [\![E_n]\!]\}$,
- $[\![E_1 + E_2]\!] = [\![E_1]\!] \cup [\![E_2]\!]$,
- $[\![E_1 ._\Box E_2]\!] = [\![E_1]\!]\{\Box \leftarrow [\![E_2]\!]\}$,
- $[\![E^{*,\Box}]\!] = [\![E]\!]^{*,\Box}$

Example 19. Let $\mathcal{F} = \{0, nil, s(), cons(,)\}$ and $\Box \in \mathcal{K}$ then

$$nil + (cons(0, \Box)^{*,\Box}) ._\Box nil$$

is a regular expression of $Regexp(\mathcal{F}, \mathcal{K})$ which denotes the set of lists of zeros:

$$\{nil, cons(0, nil), cons(0, cons(0, nil)), \dots\}$$

In the remaining of this section, we compare the relative expressive power of regular expressions and regular languages. It is easy to prove that for each regular expression E, the set $[\![E]\!]$ is a regular tree language. The proof is done by structural induction on E. The first three cases are obvious and the two last cases are consequences of Propositions 5 and 4. The converse, i.e. a regular tree language can be denoted by a regular expression, is more involved and the proof is similar to the proof of Kleene's theorem for word language. Let us state the result first.

Proposition 6. *Let $\mathcal{A} = (Q, \mathcal{F}, Q_F, \Delta)$ be a bottom-up tree automaton, then there exists a regular expression E of $Regexp(\mathcal{F}, Q)$ such that $L(\mathcal{A}) = [\![E]\!]$.*

The occurrence of symbols of Q in the regular expression denoting $L(\mathcal{A})$ doesn't cause any trouble since we have seen that a regular expression of $Regexp(\mathcal{F}, Q)$ can denote a language of $T_\mathcal{F}$.

Proof. The proof is similar to the proof for word languages and word automata. For each $1 \leq i, j, \leq |Q|, K \subseteq Q$, we define the set $T(i, j, K)$ as the set of trees t of $T(\mathcal{F} \cup K)$ such that

- if an element of K appears in t then it is a leaf,
- each leaf of t is in K or no leaf is in K,
- there is a run r of $\mathcal{A}$ such that
  - $r(\epsilon) = q_i$,
  - for each $p \in \mathcal{P}os(()t)$ such that p is not a position of leaf nor of the root, $r(p) \in \{q_1, \dots, q_j\}$.

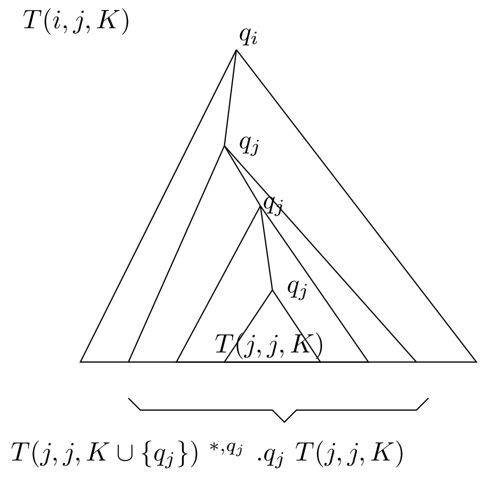
Roughly speaking, a term is in $T(i, j, K)$ if we can reach q_i at the root by using only states in $\{q_1, \dots, q_j\}$ when we assume that the leaves are states of K. By definition, $L(\mathcal{A})$ the language accepted by $\mathcal{A}$ is the union of the $T(i, |Q|, \emptyset)$'s for i such that q_i is a final state: these terms are the terms of $T(\mathcal{F})$ such that there is a successful run using any possible state of Q. Now, we prove by induction on j that $T(i, j, K)$ can be denoted by a regular expression of $Regexp(\mathcal{F}, Q)$.

- Base case $j = 0$. The set $T(i, 0, K)$ is the set of trees t where the root is labelled by q_i, the leaves are in $\mathcal{F} \cup K$ and no internal node is labelled by some q. Therefore there exist $a_1, \dots, a_n, a \in \mathcal{F} \cup K$ such that $t = f(a_1, \dots, a_n)$ or $t = a$, hence $T(i, j, \emptyset)$ is finite and can be denoted by a regular expression of $Regexp(\mathcal{F} \cup Q)$.
- Induction case. Let us assume that for any $i', K' \subseteq Q$ and $0 \leq j' < j$, the set $T(i', j', K')$ can be denoted by a regular expression. We can write the following equality:

$$\begin{aligned} T(i,j,K) = \; & T(i,j-1,K) \\ & \cup \\ & T(i,j-1,K \cup \{q_j\}) \; .q_j \; T(j,j,K \cup \{q_j\})^{\;*,q_j} \; .q_j \; T(j,j,K) \end{aligned}$$

The inclusion of $T(i,j,K)$ in the right-hand side of the equality can be easily seen from the next picture.

Decomposition of a term of $T(i,j,K)$



The converse inclusion is also not difficult. By definition, we have that:
$T(i,j-1,K) \subseteq T(i,j,K)$
and we can easily prove by induction on the number of occurrences of q_j that:
$T(i,j-1,K \cup \{q_j\}) \; .q_j \; T(j,j-1,K \cup \{q_j\})^{\;*,q_j} \; .q_j \; T(j,j-1,K) \subseteq T(i,j,K)$

By induction hypothesis, each set of the right-hand side of the equality defining $T(i,j,K)$ can be denoted by a regular expression of $Regex(\mathcal{F} \cup Q)$. This yields the desired result because the union of these sets is represented by the sum of the corresponding expressions.

□

Since we have already seen that regular expressions denote recognizable tree languages and that recognizable languages are regular, we can state Kleene's theorem for tree languages.

Theorem 16. *A tree language is recognizable if and only if it can be denoted by a regular tree expression.*

2.3 Regular equations.

Looking at our example of the set of lists of natural numbers, we can realize that these lists can be defined by equations instead of grammar rules. For instance, denoting set union by +, we could replace the grammar given in Section 2.1.1 by the following equations.

$$\begin{array}{l} Nat = 0 \ + \ s(Nat) \\ List = nil \ + \ cons(Nat, List) \end{array}$$

where the variables are *List* and *Nat*. To get the usual lists of nonnegative numbers, we must restrict ourselves to the least fixed-point solution of this set of equations. Systems of language equations do not always have solution nor does a least solution always exist. Therefore we shall study ***regular equation systems*** defined as follows.

Definition 3. *Let $X_1, \ldots, X_n$ be variables denoting sets of trees, let s_i^j be terms over $\mathcal{F} \cup \{X_1, \ldots, X_n\}$ for $1 \leq j \leq n_i$, $1 \leq i \leq n$, then a regular equation system S is a set of equations of the form:*

$$\begin{array}{rcl} X_1 & = & s_1^1 + \ldots + s_{n_1}^1 \\ & & \ldots \\ X_p & = & s_1^p + \ldots + s_{n_p}^p \end{array}$$

A solution of S is any n-tuple $(L_1, \ldots, L_n)$ of languages of $T_{\mathcal{F}}$ such that

$$\begin{array}{rcl} L_1 & = & s_1^1\{X_1 \leftarrow L_1, \ldots, X_n \leftarrow L_n\} \cup \ldots \cup s_{n_1}^1\{X_1 \leftarrow L_1, \ldots, X_n \leftarrow L_n\} \\ & & \ldots \\ L_p & = & s_1^p\{X_1 \leftarrow L_1, \ldots, X_n \leftarrow L_n\} \cup \ldots \cup s_{n_p}^p\{X_1 \leftarrow L_1, \ldots, X_n \leftarrow L_n\} \end{array}$$

We define the ordering $\subseteq$ on $T(\mathcal{F}) \times \ldots \times T(\mathcal{F})$ by $(L_1, \ldots, L_n) \subseteq (L'_1, \ldots, L'_n)$ if and only if $L_i \subseteq L'_i$ for all $i = 1, \ldots, n$. From the definition, we have that $(\emptyset, \ldots, \emptyset)$ is the smallest element of $\subseteq$ and that each increasing sequence has an upper bound. The fixed-point operator $\mathcal{TS}$ associated with an equation system S is an operator from $(T(\mathcal{F}), \ldots, T(\mathcal{F}))$ into $(T(\mathcal{F}), \ldots, T(\mathcal{F}))$ such that

$$\begin{array}{l} \mathcal{TS}(L_1, \ldots, L_n) = (L'_1, \ldots, L'_n) \\ \quad where \\ \begin{array}{rcl} L'_1 & = & L_1 \cup s_1^1\{X_1 \leftarrow L_1, \ldots, X_n \leftarrow L_n\} \cup \ldots \cup s_{n_1}^1\{X_1 \leftarrow L_1, \ldots, X_n \leftarrow L_n\} \\ & & \ldots \\ L'_p & = & L_p \cup s_1^p\{X_1 \leftarrow L_1, \ldots, X_n \leftarrow L_n\} \cup \ldots \cup s_{n_p}^p\{X_1 \leftarrow L_1, \ldots, X_n \leftarrow L_n\} \end{array} \end{array}$$

Example 20. Let S be

$$\begin{array}{rcl} Nat & = & 0 + s(Nat) \\ List & = & nil + cons(Nat, List) \end{array}$$

then

$$\begin{array}{l} \mathcal{TS}(\emptyset, \emptyset) = (\{0\}, \{nil\}) \\ \mathcal{TS}^2(\emptyset, \emptyset) = (\{0, s(0)\}, \{nil, cons(0, nil)\}) \end{array}$$

Using a classical approach we use the fixed-point operator to compute the least fixed-point solution of a system of equations.

Proposition 7. *The fixed-point operator $\mathcal{TS}$ is continuous and its least fixed-point $\mathcal{TS}^\omega(\emptyset, \dots, \emptyset)$ is the least solution of S.*

Proof. We show that $\mathcal{TS}$ is continuous in order to use Knaster-Tarski's theorem on continuous operators. By construction, $\mathcal{TS}$ is monotonous, and the last point is to prove that if $S_1 \subseteq S_2 \subseteq \dots$ is an increasing sequence of n-tuples of languages, the equality $\mathcal{TS}(\bigcup_{i\geq 1} S_i) = \bigcup_{i\geq 1} \mathcal{TS}(S_i))$ holds. By definition, each S_i can be written as $(S_1^i, \dots, S_n^i)$.

- We have that $\bigcup_{i=1,\dots} \mathcal{TS}(S_i) \subseteq \mathcal{TS}(\bigcup_{i=1,\dots}(S_i))$ holds since the sequence $S_1 \subseteq S_2 \subseteq \dots$ is increasing and the operator $\mathcal{TS}$ is monotonous.
- Conversely we have to prove that $\mathcal{TS}(\bigcup_{i=1,\dots}(S_i)) \subseteq \bigcup_{i=1,\dots} \mathcal{TS}(S_i))$ holds. Let $(U_1, \dots, U_n) \in \bigcup_{i=1,\dots} S_i$, then there exists $i_1, \dots, i_n$ such that $U_1 \subseteq S_1^{i_1}, \dots, U_n \subseteq S_n^{i_n}$, therefore $(U_1, \dots, U_n) \subseteq (S_1^N, \dots, S_n^N)$ for $N = Max\{i_p \mid p = 1, \dots, n\}$ since the sequence $S_1, S_2, \dots$ is increasing. Therefore $\mathcal{TS}(U_1, \dots, U_n) \subseteq \bigcup_{i=1,\dots} \mathcal{TS}(S_i))$ and we are done.

□

We have introduced systems of regular equations to get the algebraic characterization of regular tree languages stated in the following theorem.

Theorem 17. *The least fixed-point solution of a system of regular equations is a tuple of regular tree languages. Conversely each regular tree language is a component of the least solution of a system of regular equations.*

Proof. Let S be a system of regular equations, and let $G_i = (X_i, \{X_1, \dots, X_n\}, \mathcal{F}, R)$ where $R = \cup_{k=1,\dots,n}\{X_k \to s_k^1, \dots, X_k \to s_k^{j_k}\}$ if the k^{th} equation of S is $X_k = s_k^1 + \dots + s_k^{j_k}$. We show that $L(X_k)$ is the k^{th} component of $(L_1, \dots, L_n)$ the least fixed-point solution of S.

- We prove that $\mathcal{TS}^p(\emptyset, \dots, \emptyset) \subseteq (L(X_1), \dots, L(X_n))$ by induction on p. Let us assume that this property holds for all $q \leq p$. Let $(u_1, \dots, u_n) \in \mathcal{TS}^{k+1}(\emptyset, \dots, \emptyset)$, then for each i there is some s_i^j such that $u_i = s_{i_j}\{X_1 \to v_1, \dots, X_n \to v_n\}$ with $(v_1, \dots, v_n) \in \mathcal{TS}^k(\emptyset, \dots, \emptyset)$, therefore $u_i \in L(X_i)$ using the induction hypothesis.
- We prove now that $(L(X_1), \dots, L(X_n)) \subseteq \mathcal{TS}^\omega(\emptyset, \dots, \emptyset)$ by induction on derivation length. Let us assume that for each i, for each $m \leq p$, the implication $X_i \to^q u_i$ implies that $u_i \in \mathcal{TS}^m(\emptyset, \dots, \emptyset)$ holds. Let $X_i \to^{p+1} v_i$, then $v_i = s_i^j(u_1, \dots, u_n)$ with $X_j \to^m u_j$ for some $m \leq p$. By induction hypothesis $u_j \in \mathcal{TS}^m(\emptyset, \dots, \emptyset) \subseteq \mathcal{TS}^k(\emptyset, \dots, \emptyset)$ and we are done.

Conversely, given a regular grammar $G = (X, \{X_1, \dots, X_n\}, \mathcal{F}, R)$, with $R = \{X_1 \to s_1^1, \dots, X_1 \to s_{p_1}^1, \dots, X_n \to s_1^n, \dots, X_n \to s_{p_n}^n\}$, a similar proof yields that the least solution of the system

$$\begin{array}{rcl} X_1 & = & s_1^1 + \dots + s_{p_1}^1 \\ & & \dots \\ X_n & = & s_1^n + \dots + s_{p_n}^n \end{array}$$

is $(L(X_1), \dots, L(X_n))$. □

Example 21. The grammar with axiom *List*, non-terminals *List*, *Nat* terminals $0, s(), nil, cons(,)$ and rules

$$\begin{array}{lcl} List & \rightarrow & nil \\ List & \rightarrow & cons(Nat, List) \\ Nat & \rightarrow & 0 \\ Nat & \rightarrow & s(Nat) \end{array}$$

generates the second component of the least solution of the system given in the previous example.

2.4 Context-free word languages and regular tree languages

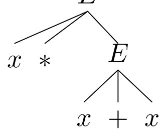
Context-free word languages and regular tree languages are strongly related. This is not surprising since derivation trees of context-free languages and derivations of tree grammars look alike. For instance let us consider the context-free language of arithmetic expressions on +,∗ and a variable x. A context-free word grammar generating this set is $E \rightarrow x \mid E + E \mid E * E$ where E is the axiom. The generation of a word from the axiom can be described by a derivation tree which has the axiom at the root and where the generated word can be read by picking up the leaves of the tree from the left to the right (computing what we call the yield of the tree). The rules for constructing derivation trees show some regularity, which suggests that this set of trees is regular. The aim of this section is to show that this is true indeed. However, there are some traps which must be avoided when linking tree and word languages. First, we describe how to relate word and trees. The symbols of $\mathcal{F}$ are used to build trees but also words (by taking a symbol of $\mathcal{F}$ as a letter). The ***Yield*** operator computes a word from a tree by concatenating the leaves of the tree from the left to the right. More precisely, it is defined as follows.

$$Yield(a) = a$$

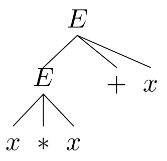
$$Yield(f(s_1, \ldots, s_n)) = Yield(s_1) \ldots Yield(s_n)$$

Example 22. Let $\mathcal{F} = \{x, +, *, E(,,)\}$ and let

$s =$



then $Yield(s) = x * x + x$ which is a word on $\{x, *, +\}$. Note that∗ and + are not the usual binary operator but syntactical symbols of arity 0. If

$t =$ 

then $Yield(t) = x * x + x$ also.

We recall that a ***context-free word grammar*** G is a tuple (A, N, T, R) where A is the axiom, N the set of non-terminals letters, T the set of terminal letters, R the set of production rules of the form $X \to \alpha$ with $X \in N, \alpha \in (T \cup N)^*$. The usual definition of derivation trees of context free word languages allow nodes labelled by a non-terminal X to have a variable number of sons, which is equal to the length of the right-hand side α of the rule $X \to \alpha$ used to build the derivation tree at this node.

Since tree languages are defined for signatures where each symbol has a fixed arity, we introduce a new symbol (X, m) for each $X \in N$ such that there is a rule $X \to \alpha$ with α of length m. Let $\mathcal{G}$ be the set composed of these new symbols and of the symbols of T. The set of derivation trees issued from $a \in \mathcal{G}$, denoted by $D(G, a)$ is the smallest set such that:

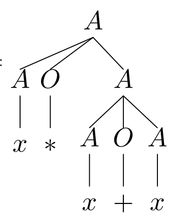
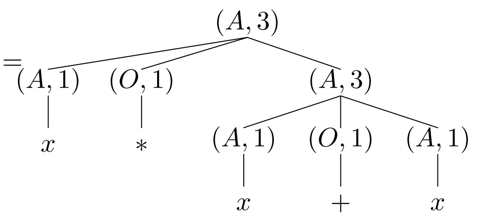
- $D(G, a) = \{a\}$ if $a \in T$,
- $(a, 0) \in D(G, a)$ if $a \to \epsilon \in R$ where ϵ is the empty word,
- $(a, p)(t_1, \ldots, t_p) \in D(G, (a, p))$ if $t_1 \in D(G, a_1), \ldots, t_p \in D(G, a_p)$ and $(a \to a_1 \ldots a_p) \in R$ where $a_i \in \mathcal{G}$.

The set of ***derivation trees*** of G is $D(G) = \cup_{(A,i) \in \mathcal{G}} D(G, (A, i))$.

Example 23. Let $T = \{x, +, *\}$ and let G be the context free word grammar with axiom A, non terminal O, and rules

$$\begin{array}{l} A \to AOA \\ A \to x \\ O \to + \\ O \to * \end{array}$$

Let the word $u = x * x + x$, a derivation tree for u with G is $d(u, G)$, and the same derivation tree with our notations is $D(u, G)$

$d(u, G) =$  ; $D(u, G) =$ 

By definition, the language generated by a context-free word grammar G is the set of words computed by applying the *Yield* operator to derivation trees of G. The next theorem states how context-free word languages and regular tree languages are related.

Theorem 18. *The following statement holds.*

1. *Let G be a context-free word grammar, then the set of derivation trees of $L(G)$ is a regular tree language.*

2. *Let L be a regular tree language then $Yield(L)$ is a context-free word language.*

3. *There exists a regular tree language which is not the set of derivation trees of a context-free language.*

Proof. We give the proofs of the three statements.

1. Let $G = (A, N, T, R)$ be a context-free word language. We consider the tree grammar $G' = (A, N, \mathcal{F}, R'))$ such that

   - the axiom and the set of non-terminal symbols of G and G' are the same,
   - $\mathcal{F} = T \cup \{(X, n) \mid X \in N,\ \exists X \rightarrow \alpha \in R \text{ with } \alpha \text{ of length } n\}$,
   - if $(X \rightarrow a_1 \dots a_p) \in R$ then $(X \rightarrow (X, p)(a_1, \dots, a_p)) \in R'$

   Then $L(G) = \{\mathit{Yield}(s) \mid s \in L(G')\}$. The proof is a standard induction on derivation length. It is interesting to remark that there may and usually does exist several tree languages (not necessarily regular) such that the corresponding word language obtained via the *Yield* operator is a given context-free word language.

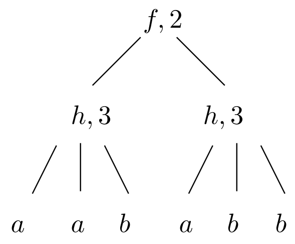
2. Let G be a normalized tree grammar (A, X, N, R). We build the word context-free grammar $G' = (A, X, N, R')$ such that a rule $X \rightarrow X_1 \dots X_n$ (resp. $X \rightarrow a$) is in R' if and only if the rule $X \rightarrow f(X_1, \dots, X_n)$ (resp. $X \rightarrow a$) is in R for some f. It is straightforward to prove by induction on the length of derivation that $L(G') = \mathit{Yield}(L(G))$.

3. Let G be the regular tree grammar with axiom X, non-terminals X, Y, Z, terminals a, b, f, h and rules

$$\begin{array}{lcl} X & \rightarrow & f(Y, Z) \\ Y & \rightarrow & h(a, Y, b) \\ Y & \rightarrow & a \\ Z & \rightarrow & h(a, Z, b) \\ Z & \rightarrow & b \end{array}$$

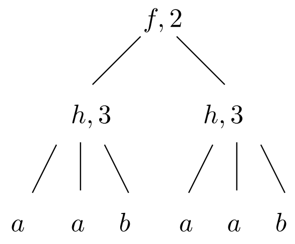
   Assume that $L(G)$ is the set of derivation trees of some context-free word grammar. The following tree is in $L(G)$ (arity have been indicated explicitly to make the link with derivation trees):

$f(h(a,a,b),h(a,b,b))$



To generate the first node one must have a rule $F \to HH$ where F is the axiom and rules $H \to aab$, $H \to abb$ (to get the inner nodes). Therefore the following tree:

$f(h(a,a,b),h(a,a,b))$



should be in $L(G)$ which is not the case.

□

2.5 Beyond regular tree languages: context-free tree languages

For word language, the story doesn't end with regular languages but there is a strict hierarchy.

$$\text{regular} \subset \text{context-free} \subset \text{recursively enumerable}$$

Recursively enumerable tree languages are languages generated by tree grammar as defined in the beginning of the chapter, and this class is far too general for having good properties. Actually, any Turing machine can be simulated by a one rule rewrite system which shows how powerful tree grammars are (any grammar rule can be seen as a rewrite rule by considering both terminals and non-terminals as syntactical symbols). Therefore, most of the research has been done on context-free tree languages which we describe now.

2.5.1 Context-free tree languages

A ***context-free tree grammar*** is a tree grammar $G = (A, N, \mathcal{F}, R)$ where the rules have the form $X(x_1, \dots, x_n) \to t$ with t a tree of $T(\mathcal{F} \cup N \cup \{x_1, \dots, x_n\})$,

$x_1, \dots, x_n \in \mathcal{X}$ where $\mathcal{X}$ is a set of reserved variables with $\mathcal{X} \cap (\mathcal{F} \cup N) = \emptyset$, X a non-terminal of arity n. The definition of the derivation relation is slightly more complicated than for regular tree grammar: a term t derives a term t' if no variable of $\mathcal{X}$ occurs in t or t', there is a rule $l \rightarrow r$ of the grammar, a substitution σ such that the domain of σ is included in $\mathcal{X}$ and a context C such that $t = C[l\sigma]$ and $t' = C[r\sigma]$. The ***context-free tree language*** $L(G)$ is the set of trees which can be derived from the axiom of the context-free tree grammar G.

Example 24. The grammar of axiom *Prog*, set of non-terminals $\{Prog, Nat, Fact()\}$, set of terminals $\{0, s, cond(,,), eq(,), times(,), dec()\}$ and rules

$$\begin{array}{lll} Prog & \rightarrow & Fact(Nat) \\ Nat & \rightarrow & 0 \\ Nat & \rightarrow & s(Nat) \\ Fact(x) & \rightarrow & cond(eq(x,0), s(0), times(x, Fact(dec(x)))) \end{array}$$

where $\mathcal{X} = \{x\}$ is a context-free tree grammar. The reader can easily see that the last rule is the classical definition of the factorial function.

The derivation relation associated to a context-free tree grammar G is a generalization of the derivation relation for regular tree grammar. The derivation relation $\rightarrow$ is a relation on pairs of terms of $T(\mathcal{F} \cup N)$ such that $s \rightarrow t$ iff there is a rule $X(x_1, \dots, x_n) \rightarrow \alpha \in R$, a context C such that $s = C[X(t_1, \dots, t_n)]$ and $t = C[\alpha\{x_1 \leftarrow t_1, \dots, x_n \leftarrow t_n\}]$. The language generated by G, denoted by $L(G)$ is the set of terms of $T(\mathcal{F})$ which can be reached by successive derivations starting from the axiom. Such languages are called ***c***ontext-free tree languages. Context-free languages are closed under union, concatenation and closure.

2.5.2 IO and OI tree grammars

Context-free tree grammars have been extensively studied in connection with the theory of recursive program scheme. A non-terminal F can be seen as a function name and production rules $F(x_1, \dots, x_n) \rightarrow t$ define the function. Recursive definitions are allowed since t may contain occurrences of F. Since we know that such recursive definitions may not give the same results depending on the evaluation strategy, IO and OI tree grammars have been introduced to account for such differences.

A context-free grammar is ***IO*** (for innermost-outermost) if we restrict legal derivations to derivations where the innermost terminals are derived first. This control corresponds to call by value evaluation. A context-free grammar is ***OI*** (for outermost-innermost) if we restrict legal derivations to derivations where the outermost terminals are derived first. This corresponds to call by name evaluation. Therefore, given one context-free grammar G, we can define *IO-G* and *OI-G* and the next example shows that the languages generated by these grammars may be different.

Example 25. Let G be the context-free grammar with axiom *Exp*, non-terminals $\{Exp, Nat, Dup\}$, terminals $\{double, s, 0\}$) and rules

$Exp \to Dup(Nat)$
$Nat \to s(Nat)$
$Nat \to 0$
$Dup(x) \to double(x, x)$
Then outermost-innermost derivations have the form

$$Exp \to Dup(Nat) \to double(Nat, Nat) \xrightarrow{*} double(s^n(0), s^m(0))$$

while innermost-outermost derivations have the form

$$Exp \to Dup(Nat) \xrightarrow{*} Dup(s^n(0)) \to double(s^n(0), s^n(0))$$

Therefore $L(OI\text{-}G) = \{double(s^n(0), s^m(0)) \mid n, m \in \mathbb{N}\}$ and
$L(IO\text{-}G) = \{double(s^n(0), s^n(0)) \mid n \in \mathbb{N}\}$.

A tree language L is *IO* if there is some context-free grammar G such that $L = L(IO\text{-}G)$. The next theorem shows the relation between $L(IO\text{-}G)$, $L(OI\text{-}G)$ and $L(G)$.

Theorem 19. *The following inclusion holds:* $L(IO\text{-}G) \subseteq L(OI\text{-}G) = L(G)$

The previous example shows that the inclusion can be strict. *IO*-languages are also closed under intersection with regular languages and union, but the closure under concatenation requires another definition of concatenation: each occurrence of a constant is replaced by the *same* term.

There is a lot of work on the extension of results on context-free word grammars and languages to context-free tree grammars and languages. Unfortunately, many constructions and theorem can't be lifted to the tree cases. Usually the failure is due to non-linearity which expresses that the same subtrees must occur at different positions in the tree. A similar phenomenon occurred when we stated results on recognizable languages and tree homomorphisms: the inverse image of a recognizable tree language by a tree homorphism is recognizable, but the assumption that the homomorphism is linear is needed to show that the direct image is recognizable.

2.6 Exercises

Exercise 19. Let $\mathcal{F} = \{f(,), g(), a\}$. Consider the automaton $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ defined by: $Q = \{q, q_g, q_f\}$, $Q_f = \{q_f\}$, and $\Delta =$

$$\begin{array}{lrcllcl} \{ & a & \to & q(a) & g(q(x)) & \to & q(g(x)) \\ & g(q(x)) & \to & q_g(g(x)) & g(q_g(x)) & \to & q_f(g(x)) \\ & f(q(x), q(y)) & \to & q(f(x, y)) & \}. & & \end{array}$$

Define a regular tree grammar generating $L(\mathcal{A})$.

Exercise 20. Let $\mathcal{F} = \{f(,), g(), a\}$. Let G be the regular tree grammar with axiom X, non-terminal A, and rules

$$X \to f(g(A), A)$$
$$A \to g(g(A))$$

Define a top-down NFTA, a NFTA and a DFTA for $L(G)$. Is it possible to define a top-down DFTA for this language?

Exercise 21. Let $\mathcal{F} = \{f(,), a\}$. Let G be the regular tree grammar with axiom X, non-terminals A, B, C and rules

$$\begin{array}{l} X \to C \\ X \to a \\ X \to A \\ A \to f(A, B) \\ B \to a \end{array}$$

Compute the reduced regular tree grammar equivalent to G applying the algorithm defined in the proof of Proposition 2. Now, consider the same algorithm, but first apply step 2 and then step 1. Is the output of this algorithm reduced ? equivalent to G ?

Exercise 22.

1. Prove Theorem 6 using regular tree grammars.
2. Prove Theorem 7 using regular tree grammars.

Exercise 23. Let $\mathcal{F}$ be a signature, let $\mathcal{X}$ be a countable set of variables, we define set expressions by the grammar:

$$sexp ::= \bot \mid \top \mid X \mid f(sexp, \dots, sexp) \mid f^{-1}_{(i)}(sepx) \mid sexp \cap sexp$$

where $X \in \mathcal{X}$, $f \in \mathcal{F}$. We denote by $\mathcal{F}'$ the signature $\bigcup_{f \in \mathcal{F}} \{f_1^{-1}, \dots, f^{-1}_{\text{Arity}f}\} \cup \{\cap, \top, \bot\} \cup \mathcal{F}$.

An interpretation I assigns a set of ground terms to each variable and it is extended to any set expression by setting:
$I(\bot) = \emptyset$
$I(\top) = T(\mathcal{F})$
$I(s_1 \cap s_2) = I(s_1) \cap I(s_2)$
$I(f(s_1, \dots, s_n)) = f(I(s_1), \dots, I(s_n))$
$I(f^{-1}_{(i)}(s)) = \{u_i \in T(\mathcal{F}) \mid \exists u_1, \dots, u_n \in T(\mathcal{F}) f(u_1, \dots, u_n) \in I(s)\}$

A set expression grammar $G = (A, \mathcal{X}, \mathcal{F}, R)$ is defined as tree grammar except that production rules have the form $X \to sexp$. The language generated by a set expression grammar G is $I(G) = \{I(t) \mid A \xrightarrow{*} t \ t \ \in T(\mathcal{F}')\}$.

The goal of the exercise is to show that there exists a regular tree grammar $\mathcal{G}$ such that $L(\mathcal{G}) = I(G)$.

1. Compute a set expression grammar G_1 such that $I(G) = I(G_1)$ and all rules of G_1 have the form $X \to s_1 \cap \dots \cap s_m$ where the s_i's have the form $\alpha(X_1, \dots, X_n)$ with $\alpha \in \mathcal{F}$ or a projection.
2. Compute a set expression grammar G_2 equivalent to G_1 such that all productions of G_2 have the form $X \to s_1 \cap \dots \cap s_m$ where the s_i's have the form $f(X_1, \dots, X_n)$ with $f \in \mathcal{F}$.
3. Compute a regular tree grammar $\mathcal{G}$ equivalent to G_2.

Exercise 24. (Local languages) Let $\mathcal{F}$ be a signature, let t be a term of $T(\mathcal{F})$, then we define $fork(t)$ as follows:

- $fork(a) = \emptyset$, for each constant symbol a;
- $fork(f(t_1, \dots, t_n)) = \{f(\mathcal{H}ead(t_1), \dots, \mathcal{H}ead(t_n))\} \cup \bigcup_{i=1}^{i=n} fork(t_i)$

A tree language L is ***local*** if and only if there exist a set $\mathcal{F}' \subseteq \mathcal{F}$ and a set $G \subseteq fork(T(\mathcal{F}))$ such that $t \in L$ iff $root(t) \in \mathcal{F}'$ and $fork(t) \subseteq G$. Prove that every local tree language is a regular tree language. Prove that a language is local iff it is the set of derivation trees of a context-free word language.

Exercise 25. The pumping lemma for context-free word languages states:

> for each context-free language L, there is some constant $k \geq 1$ such that each $z \in L$ of length greater than or equal to k can be written $z = uvwxy$ such that vx is not the empty word, vwx has length less than or equal to k, and for each $n \geq 0$, the word uv^nwx^ny is in L.

Prove this result using the pumping lemma for tree languages and the results of this chapter.

Exercise 26. Let $\mathcal{F}$ be a ranked alphabet, let t be a term of $T(\mathcal{F})$, then we define the word language $Branch(t)$ as follows:

- $Branch(a) = a$, for each constant symbol a;
- $Branch(f(t_1, \ldots, t_n)) = \bigcup_{i=1}^{i=n}\{fu \mid u \in Branch(t_i)\}$

Let L be a regular tree language, prove that $Branch(L) = \bigcup_{t\in L} Branch(t)$ is a regular word language. What about the converse?

Exercise 27. Show that a tree language L such that $Yield(L)$ is a regular word language is a regular tree language.

Exercise 28.

1. Let $\mathcal{F}$ be a ranked alphabet such that $\mathcal{F}_0 = \{a, b\}$. Find a regular tree language L such that $Yield(L) = \{a^nb^n \mid n \geq 0\}$. Find a non regular tree language L such that $Yield(L) = \{a^nb^n \mid n \geq 0\}$.
2. Same questions with $Yield(L) = \{u \in \mathcal{F}_0^* \mid |u|_a = |u|_b\}$ where $|u|_a$ (respectively $|u|_b$) denotes the number of a (respectively the number of b) in u.
3. Let $\mathcal{F}$ be a ranked alphabet such that $\mathcal{F}_0 = \{a, b, c\}$, let $A_1 = \{a^nb^nc^p \mid n, p \geq 0\}$, and let $A_2 = \{a^nb^pc^p \mid n, p \geq 0\}$. Find regular tree languages such that $Yield(L_1) = A_1$ and $Yield(L_2) = A_2$. Does there exist a regular tree language such that $Yield(L) = A_1 \cap A_2$.

Exercise 29.

1. Let G be the context free word grammar with axiom X, terminals a, b, and rules

$$\begin{array}{l} X \rightarrow XX \\ X \rightarrow aXb \\ X \rightarrow \epsilon \end{array}$$

   where ϵ stands for the empty wlord. What is the word language $L(G)$? Give a derivation tree lfor $u = aabbab$.

2. Let G' be the context frlee word grammar in Greibach normal form with axiom X, non terminlals X', Y', Z' terminals a, b, and rules l

$$\begin{array}{l} X' \rightarrow aX'Y' \\ X' \rightarrow aY' \\ X' \rightarrow aX'Z' \\ X' \rightarrow aZ' \\ Y' \rightarrow bX' \\ Z' \rightarrow b \end{array}$$

prove that $L(G') = L(G)$. Give a derivation tree for $u = aabbab$.

3. Find a context free word grammar G'' such that $L(G'') = A_1 \cup A_2$ (A_1 and A_2 are defined in Exercise 28). Give two derivation trees for $u = abc$.

Exercise 30. Let $\mathcal{F}$ be a ranked alphabet.

1. Let L and L' be two regular tree languages. Compare the sets $\mathit{Yield}(L \cap L')$ and $\mathit{Yield}(L) \cap \mathit{Yield}(L')$.
2. Let A be a subset of $\mathcal{F}_0$. Prove that $T(\mathcal{F}, A) = T(\mathcal{F} \cap A)$ is a regular tree language. Let L be a regular tree language over $\mathcal{F}$, compare the sets $\mathit{Yield}(L \cap T(\mathcal{F}, A))$ and $\mathit{Yield}(L) \cap \mathit{Yield}(T(\mathcal{F}, A))$.
3. Let R be a regular word language over $\mathcal{F}_0$. Let $T(\mathcal{F}, R) = \{t \in T(\mathcal{F}) \mid \mathit{Yield}(t) \in R\}$. Prove that $T(\mathcal{F}, R)$ is a regular tree language. Let L be a regular tree language over $\mathcal{F}$, compare the sets $\mathit{Yield}(L \cap T(\mathcal{F}, R))$ and $\mathit{Yield}(L) \cap \mathit{Yield}(T(\mathcal{F}, R))$. As a consequence of the results obtained in the present exercise, what could be said about the intersection of a context free word language and of a regular tree language?

2.7 Bibliographic notes

This chapter only scratches the topic of tree grammars and related topics. A useful reference on algebraic aspects of regular tree language is [GS84] which contains a lot of classical results on these features. There is a huge litterature on tree grammars and related topics, which is also relevant for the chapter on tree transducers, see the references given in this chapter. The reader may consult [Eng82, ES78] for IO and OI grammars. The connection between recursive program scheme and formalisms for regular tree languages is also well-known, see [Cou86] for instance. We should mention that some open problems like equivalence of deterministic tree grammars are now solved using the result of Senizergues on the equivalence of deterministic pushdown word automata [Sén97].

Chapter 3

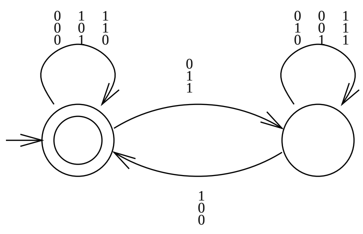
Automata and n-ary relations

3.1 Introduction

As early as in the 50s, automata, and in particular tree automata, played an important role in the development of verification . Several well-known logicians, such as A. Church, J.R. Büchi, Elgott, MacNaughton, M. Rabin and others contributed to what is called “the trinity” by Trakhtenbrot: Logic, Automata and Verification (of Boolean circuits).

The idea is simple: given a formula ϕ with free variables $x_1, ..., x_n$ and a domain of interpretation D, ϕ defines the subset of D^n containing all assignments of the free variables $x_1, \ldots, x_n$ that satisfy ϕ. Hence formulas in this case are just a way of defining subsets of D^n (also called n-ary relations on D). In case $n = 1$ (and, as we will see, also for $n > 1$), finite automata provide another way of defining subsets of D^n. In 1960, Büchi realized that these two ways of defining relations over the free monoid $\{0, \ldots, n\}^*$ coincide when the logic is the *sequential calculus*, also called *weak second-order monadic logic with one successor*, WS1S. This result was extended to tree automata: Doner, Thatcher and Wright showed that the definability in the weak second-order monadic logic with k successors, WSkS coincide with the recognizability by a finite tree automaton. These results imply in particular the decidability of WSkS, following the decision results on tree automata (see chapter 1).

These ideas are the basis of several decision techniques for various logics some of which will be listed in Section 3.4. In order to illustrate this correspondence, consider Presburger's arithmetic: the atomic formulas are equalities and inequalities $s = t$ or $s \geq t$ where s, t are sums of variables and constants. For instance $x+y+y = z+z+z+1+1$, also written $x+2y = 3z+2$, is an atomic formula. In other words, atomic formulas are linear Diophantine (in)equations. Then atomic formulas can be combined using any logical connectives among $\wedge, \vee, \neg$ and quantifications $\forall, \exists$. For instance $\forall x (\forall y . \neg(x = 2y)) \Rightarrow (\exists y . x = 2y + 1))$ is a (true) formula of Presburger's arithmetic. Formulas are interpreted in the natural numbers (non-negative integers), each symbol having its expected meaning. A *solution* of a formula $\phi(x)$ whose only free variable is x, is an assignment of x to a natural number n such that $\phi(n)$ holds true in the interpretation. For



$\begin{smallmatrix}0\\0\\0\end{smallmatrix}$ $\begin{smallmatrix}1\\0\\1\end{smallmatrix}$ $\begin{smallmatrix}1\\1\\0\end{smallmatrix}$
$\begin{smallmatrix}0\\1\\0\end{smallmatrix}$ $\begin{smallmatrix}0\\0\\1\end{smallmatrix}$ $\begin{smallmatrix}1\\1\\1\end{smallmatrix}$
$\begin{smallmatrix}0\\1\\1\end{smallmatrix}$
$\begin{smallmatrix}1\\0\\0\end{smallmatrix}$

Figure 3.1: The automaton with accepts the solutions of $x = y + z$

instance, if $\phi(x)$ is the formula $\exists y. x = 2y$, its solutions are the even numbers.

Writing integers in base 2, they can be viewed as elements of the free monoid $\{0, 1\}^*$, i.e. words of 0s and 1s. The representation of a natural number is not unique as $01 = 1$, for instance. Tuples of natural numbers are displayed by stacking their representations in base 2 and aligning on the right, then completing with some 0s on the left in order to get a rectangle of bits. For instance the pair (13,6) is represented as $\begin{smallmatrix}1&1&0&1\\0&1&1&0\end{smallmatrix}$ (or $\begin{smallmatrix}0&1&1&0&1\\0&0&1&1&0\end{smallmatrix}$ as well). Hence, we can see the solutions of a formula as a subset of $(\{0, 1\}^n)^*$ where n is the number of free variables of the formula.

It is not difficult to see that the set of solutions of any atomic formula is recognized by a finite word automaton working on the alphabet $\{0, 1\}^n$. For instance, the solutions of $x = y + z$ are recognized by the automaton of Figure 3.1.

Then, and that is probably one of the key ideas, each logical connective corresponds to a basic operation on automata (here word automata): $\vee$ is a union, $\wedge$ and intersection, $\neg$ a complement, $\exists x$ a *projection* (an operation which will be defined in Section 3.2.4). It follows that the set of solutions of any Presburger formula is recognized by a finite automaton.

In particular, a closed formula (without free variable), holds true in the interpretation if the initial state of the automaton is also final. It holds false otherwise. Therefore, this gives both a decision technique for Presburger formulas by computing automata and an effective representation of the set of solutions for open formulas.

The example of Presburger's arithmetic we just sketched is not isolated. That is one of the purposes of this chapter to show how to relate finite tree automata and formulas.

In general, the problem with these techniques is to design an appropriate notion of automaton, which is able to recognize the solutions of atomic formulas and which has the desired closure and decision properties. We have to cite here the famous *Rabin automata* which work on infinite trees and which have indeed the closure and decidability properties, allowing to decide the full second-order monadic logic with k successors (a result due to M. Rabin, 1969). It is however out of the scope of this book to survey automata techniques in logic and computer science. We restrict our attention to finite automata on finite trees and refer to the excellent surveys [Rab77, Tho90] for more details on other applications of automata to logic.

We start this chapter by reviewing some possible definitions of automata on pairs (or, more generally, tuples) of finite trees in Section 3.2. We define in this way several notions of recognizability for relations, which are not necessary unary, extending the frame of chapter 1. This extension is necessary since, automata recognizing the solutions of formulas actually recognize n-tuples of solutions, if there are n free variables in the formula.

The most natural way of defining a notion of recognizability on tuples is to consider products of recognizable sets. Though this happens to be sometimes sufficient, this notion is often too weak. For instance the example of Figure 3.1 could not be defined as a product of recognizable sets. Rather, we stacked the words and recognized these codings. Such a construction can be generalized to trees (we have to overlap instead of stacking) and gives rise to a second notion of recognizability. We will also introduce a third class called "Ground Tree Transducers" which is weaker than the second class above but enjoys stronger closure properties, for instance by iteration. Its usefulness will become evident in Section 3.4.

Next, in Section 3.3, we introduce the weak second-order monadic logic with k successor and show Thatcher and Wright's theorem which relates this logic with finite tree automata. This is a modest insight into the relations between logic and automata.

Finally in Section 3.4 we survey a number of applications, mostly issued from Term Rewriting or Constraint Solving. We do not detail this part (we give references instead). The goal is to show how the simple techniques developed before can be applied to various questions, with a special emphasis on decision problems. We consider the theories of *sort constraints* in Section 3.4.1, the theory of *linear encompassment* in Section 3.4.2, the theory of ground term rewriting in Section 3.4.3 and reduction strategies in orthogonal term rewriting in Section 3.4.4. Other examples are given as exercises in section 3.5 or considered in chapters 4 and 5.

3.2 Automata on tuples of finite trees

3.2.1 Three notions of recognizability

Let $Rec_\times$ be the subset of n-ary relations on $T(\mathcal{F})$ which are finite unions of products $S_1 \times \ldots \times S_n$ where $S_1, \ldots, S_n$ are recognizable subsets of $T(\mathcal{F})$. This notion of recognizability of pairs is the simplest one can imagine. Automata for such relations consist of pairs of tree automata which work independently. This notion is however quite weak, as e.g. the diagonal

$$\Delta = \{(t,t) \mid t \in T(\mathcal{F})\}$$

does not belong to $Rec_\times$. Actually a relation $R \in Rec_\times$ does not really relate its components !

The second notion of recognizability is used in the correspondence with WSkS and is strictly stronger than the above one. Roughly, it consists in overlapping the components of a n-tuple, yielding a term on a product alphabet. Then define *Rec* as the set of sets of pairs of terms whose overlapping coding is recognized by a tree automaton on the product alphabet.

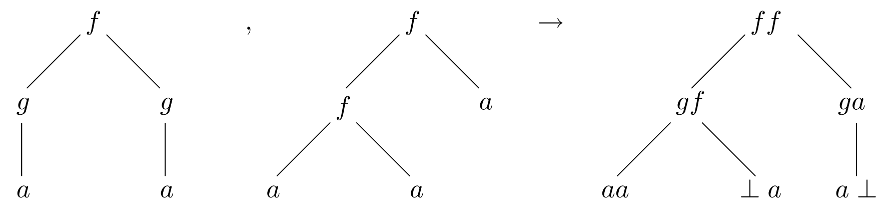


Figure 3.2: The overlap of two terms

Let us first define more precisely the notion of "coding". (This is illustrated by an example on Figure 3.2). We let $\mathcal{F}' = (\mathcal{F} \cup \{\perp\})^n$, where $\perp$ is a new symbol. This is the idea of "stacking" the symbols, as in the introductory example of Presburger's arithmetic. Let k be the maximal arity of a function symbol in $\mathcal{F}$. Assuming $\perp$ has arity 0, the arities of function symbols in $\mathcal{F}'$ are defined by $a(f_1 \dots f_n) = \max(a(f_1), \dots, a(f_n))$.

The coding of two terms $t_1, t_2 \in T(\mathcal{F})$ is defined by induction:

$$[f(t_1, \dots, t_n), g(u_1, \dots, u_m)] \stackrel{\text{def}}{=} fg([t_1, u_1], \dots [t_m, u_m], [t_{m+1}, \perp], \dots, [t_n, \perp])$$

if $n \geq m$ and

$$[f(t_1, \dots, t_n), g(u_1, \dots, u_m)] \stackrel{\text{def}}{=} fg([t_1, u_1], \dots [t_n, u_n], [\perp, u_{n+1}], \dots, [\perp, u_m])$$

if $m \geq n$.

More generally, the coding of n terms $f_1(t_1^1, \dots, t_1^{k_1}), \dots, f_n(t_1, \dots, t_n^{k_n})$ is defined as

$$f_1 \dots f_n([t_1^1, \dots, t_n^1], \dots, [t_1^m, \dots, t_n^m])$$

where m is the maximal arity of $f_1, \dots, f_n \in \mathcal{F}$ and t_i^j is, by convention, $\perp$ when $j > k_i$.

Definition 4. *Rec is the set of relations $R \subseteq T(\mathcal{F})^n$ such that*

$$\{[t_1, \dots, t_n] \mid (t_1, \dots, t_n) \in R\}$$

is recognized by a finite tree automaton on the alphabet $\mathcal{F}' = (\mathcal{F} \cup \{\perp\})^n$.

For example, consider the diagonal Δ, it is in *Rec* since its coding is recognized by the bottom-up tree automaton whose only state is q (also a final state) and transitions are the rules $ff(q, \dots, q) \rightarrow q$ for all symbols $f \in \mathcal{F}$.

One drawback of this second notion of recognizability is that it is not closed under iteration. More precisely, there is a binary relation R which belongs to *Rec* and whose transitive closure is not in *Rec* (see Section 3.5). For this reason, a third class of recognizable sets of pairs of trees was introduced: the *Ground Tree Transducers* (GTT for short) .

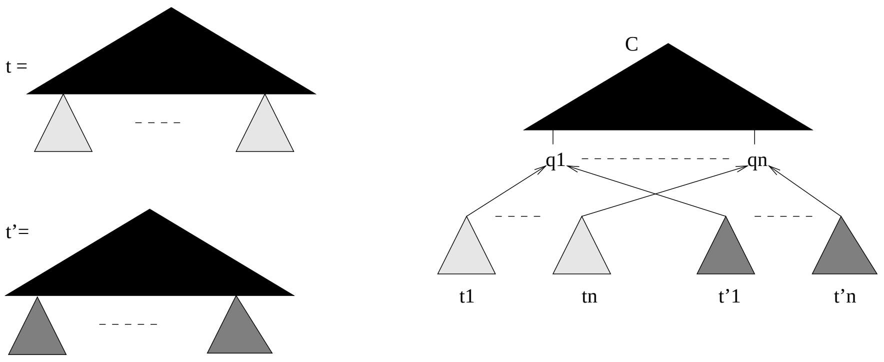


Figure 3.3: GTT acceptance

Definition 5. *A GTT is a pair of bottom-up tree automata $(\mathcal{A}_1, \mathcal{A}_2)$ working on the same alphabet. Their sets of states may however share some symbols (the* synchronization states*).*

A pair (t, t') is recognized by a GTT $(\mathcal{A}_1, \mathcal{A}_2)$ if there is a context $C \in \mathcal{C}^n(\mathcal{F})$ such that $t = C[t_1, \dots, t_n]$, $t' = C[t'_1, \dots, t'_n]$ and there are states $q_1, \dots, q_n$ of both automata such that, for all i, $t_i \xrightarrow[\mathcal{A}_1]{*} q_i$ and $t'_i \xrightarrow[\mathcal{A}_2]{*} q_i$. We write $L(\mathcal{A}_1, \mathcal{A}_2)$ the language accepted by the GTT $(\mathcal{A}_1, \mathcal{A}_2)$, i.e. the set of pairs of terms which are recognized.

The recognizability by a GTT is depicted on Figure 3.3. For instance, Δ is accepted by a GTT. Another typical example is the binary relation "one step parallel rewriting" for term rewriting system whose left members are linear and whose right hand sides are ground (see Section 3.4.3).

3.2.2 Examples of the three notions of recognizability

The first example illustrates $Rec_\times$. It will be developed in a more general framework in Section 3.4.2.

Example 26. Consider the alphabet $\mathcal{F} = \{f, g, a\}$ where f is binary, g is unary and a is a constant. Let P be the predicate which is true on t if there are terms t_1, t_2 such that $f(g(t_1), t_2)$ is a subterm of t. Then the solutions of $P(x) \wedge P(y)$ define a relation in $Rec_\times$, using twice the following automaton :

$$\begin{array}{lcl}
Q & = & \{q_f, q_g, q_\top\} \\
Q_f & = & \{q_f\} \\
T & = & \{ \qquad \begin{array}{rcl rcl}
a & \to & q_\top & f(q_\top, q_\top) & \to & q_\top \\
g(q_\top) & \to & q_\top & f(q_f, q_\top) & \to & q_f \\
g(q_f) & \to & q_f & f(q_g, q_\top) & \to & q_f \\
g(q_\top) & \to & q_g & f(q_\top, q_f) & \to & q_f\}
\end{array}
\end{array}$$

For instance the pair $(g(f(g(a), g(a))), f(g(g(a)), a))$ is accepted by the pair of automata.

The second example illustrates *Rec*. Again, it is a first account of the developments of Section 3.4.4

Example 27. Let $\mathcal{F} = \{f, g, a, \Omega\}$ where f is binary, g is unary, a and Ω are constants. Let R be the set of terms (t, u) such that u can be obtained from t by replacing each occurrence of Ω by some term in $T(\mathcal{F})$ (each occurrence of Ω needs not to be replaced with the same term). Using the notations of Chapter 2

$$R(t, u) \Longleftrightarrow u \in t.{}_{\Omega}T(\mathcal{F})$$

R is recognized by the following automaton (on codings of pairs):

$$\begin{array}{lcllllllll}
Q & = & \{q, q'\} \\
Q_f & = & \{q'\} \\
T & = & \{ & \perp a & \rightarrow & q & \perp f(q,q) & \rightarrow & q \\
 & & & \perp g(q) & \rightarrow & q & \Omega f(q,q) & \rightarrow & q' \\
 & & & \perp \Omega & \rightarrow & q & ff(q',q') & \rightarrow & q' \\
 & & & aa & \rightarrow & q' & gg(q') & \rightarrow & q' \\
 & & & \Omega\Omega & \rightarrow & q' & \Omega g(q) & \rightarrow & q' \\
 & & & \Omega a & \rightarrow & q'\}
\end{array}$$

For instance, the pair $(f(g(\Omega), g(\Omega)), f(g(g(a)), g(\Omega)))$ is accepted by the automaton: the overlap of the two terms yields

$$[tu] = ff(gg(\Omega g(\perp a)), gg(\Omega\Omega))$$

And the reduction:

$$\begin{array}{lcl}
[tu] & \xrightarrow{*} & ff(gg(\Omega g(q)), gg(q')) \\
 & \xrightarrow{*} & ff(gg(q'), q') \\
 & \rightarrow & ff(q', q') \\
 & \rightarrow & q'
\end{array}$$

The last example illustrates the recognition by a GTT. It comes from the theory of rewriting; further developments and explanations on this theory are given in Section 3.4.3.

Example 28. Let $\mathcal{F} = \{\times, +, 0, 1\}$. Let $\mathcal{R}$ be the rewrite system $0 \times x \rightarrow 0$. The many-steps reduction relation defined by $\mathcal{R}$: $\xrightarrow[\mathcal{R}]{*}$ is recognized by the GTT$(\mathcal{A}_1, \mathcal{A}_2)$ defined as follows ($+$ and $\times$ are used in infix notation to meet their usual reading):

$$\begin{array}{lclllllll}
T_1 & = & \{ & 0 & \rightarrow & q_\top & q_\top + q_\top & \rightarrow & q_\top \\
 & & & 1 & \rightarrow & q_\top & q_\top \times q_\top & \rightarrow & q_\top \\
 & & & 0 & \rightarrow & q_0 & q_0 \times q_\top & \rightarrow & q_0\} \\
T_2 & = & \{ & 0 & \rightarrow & q_0\}
\end{array}$$

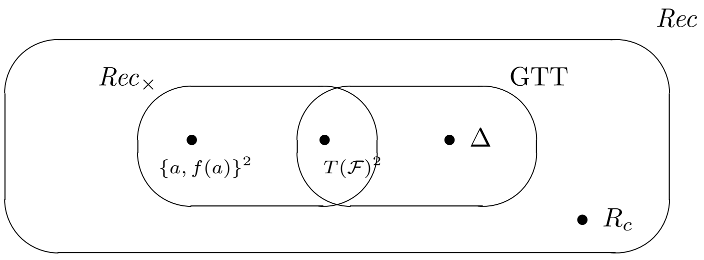


Figure 3.4: The relations between the three classes

Then, for instance, the pair $(1 + ((0 \times 1) \times 1), 1 + 0)$ is accepted by the GTT since

$$1 + ((0 \times 1) \times 1) \xrightarrow[\mathcal{A}_1]{*} 1 + (q_0 \times q_\top) \times q_\top \xrightarrow[\mathcal{A}_1]{} 1 + (q_0 \times q_\top) \xrightarrow[\mathcal{A}_1]{} 1 + q_0$$

one hand and $1 + 0 \xrightarrow[\mathcal{A}_2]{} 1 + q_0$ on the other hand.

3.2.3 Comparisons between the three classes

We study here the inclusion relations between the three classes: $Rec_\times, Rec, GTT$.

Proposition 8. *$Rec_\times \subset Rec$ and the inclusion is strict.*

Proof. To show that any relation in $Rec_\times$ is also in Rec, we have to construct from two automata $\mathcal{A}_1 = (Q_1, \mathcal{F}, Q_1^f, R_1), \mathcal{A}_2 = (\mathcal{F}, Q_2, Q_2^f, R_2)$ an automaton which recognizes the overlaps of the terms in the languages. We define such an automaton $\mathcal{A} = (Q, (\mathcal{F} \cup \{\perp\})^2, Q^f, R)$ by: $Q = (Q_1 \cup \{q_\perp\}) \times (Q_2 \cup \{q_\perp\})$, $Q^f = Q_1^f \times Q_2^f$ and R is the set of rules:

- $[f\ \perp]((q_1, q_\perp), \dots, (q_n, q_\perp)) \ \rightarrow\ (q, q_\perp)$ if $f(q_1, \dots, q_n) \rightarrow q \in R_1$
- $[\perp\ f]((q_\perp, q_1), \dots, (q_\perp, q_n)) \ \rightarrow\ (q_\perp, q)$ if $f(q_1, \dots, q_n) \rightarrow q \in R_2$
- $fg((q_1, q_1'), \dots, (q_m, q_m'), (q_{m+1}, q_\perp), \dots, (q_n, q_\perp)) \rightarrow (q, q')$ if $f(q_1, \dots, q_n) \rightarrow q \in R_1$ and $g(q_1', \dots, q_m') \rightarrow q' \in R_2$ and $n \geq m$
- $fg((q_1, q_1'), \dots, (q_n, q_n'), (q_\perp, q_{n+1}), \dots, (q_\perp, q_m)) \rightarrow (q, q')$ if $f(q_1, \dots, q_n) \rightarrow q \in R_1$ and $g(q_1', \dots, q_m') \rightarrow q' \in R_2$ and $m \geq n$

The proof that $\mathcal{A}$ indeed accepts $L(\mathcal{A}_1) \times L(\mathcal{A}_2)$ is left to the reader.

Now, the inclusion is strict since e.g. $\Delta \in Rec \setminus Rec_\times$. □

Proposition 9. *$GTT \subset Rec$ and the inclusion is strict.*

Proof. Let $(\mathcal{A}_1, \mathcal{A}_2)$ be a GTT accepting R. We have to construct an automaton $\mathcal{A}$ which accepts the codings of pairs in R.

Let $\mathcal{A}_0 = (Q_0, \mathcal{F}, Q_0^f, T_0)$ be the automaton constructed in the proof of Proposition 8. $[t,u] \xrightarrow[\mathcal{A}_0]{*} (q_1,q_2)$ if and only if $t \xrightarrow[\mathcal{A}_1]{*} q_1$ and $u \xrightarrow[\mathcal{A}_2]{*} q_2$. Now we let $\mathcal{A} = (Q_0 \cup \{q_f\}, \mathcal{F}, Q_f = \{q_f\}, T)$. T consists of T_0 plus the following rules:

$$(q,q) \quad \rightarrow \quad q_f \qquad ff(q_f,\dots,q_f) \quad \rightarrow \quad q_f$$

For every symbol $f \in F$ and every state $q \in Q_0$.

If (t,u) is accepted by the GTT, then

$$t \xrightarrow[\mathcal{A}_1]{*} C[q_1,\dots,q_n]_{p_1,\dots,p_n} \xleftarrow[\mathcal{A}_2]{*} u.$$

Then

$$[t,u] \xrightarrow[\mathcal{A}_0]{*} [C,C][(q_1,q_1),\dots,(q_n,q_n)]_{p_1,\dots,p_n} \xrightarrow[\mathcal{A}]{*} [C,C][q_f,\dots,q_f]_{p_1,\dots,p_n} \xrightarrow[\mathcal{A}]{*} q_f$$

Conversely, if $[t,u]$ is accepted by $\mathcal{A}$ then $[t,u] \xrightarrow[\mathcal{A}]{*} q_f$. By definition of $\mathcal{A}$, there should be a sequence:

$$[t,u] \xrightarrow[\mathcal{A}]{*} C[(q_1,q_1),\dots,(q_n,q_n)]_{p_1,\dots,p_n} \xrightarrow[\mathcal{A}]{*} C[q_f,\dots,q_f]_{p_1,\dots,p_n} \xrightarrow[\mathcal{A}]{*} q_f$$

Indeed, we let p_i be the positions at which one of the ϵ-transitions steps $(q,q) \rightarrow q_f$ is applied. $(n \geq 0)$. Now, $C[q_f,\dots,q_f]_{p_1,\dots,p_m} q_f$ if and only if C can be written $[C_1, C_1]$ (the proof is left to the reader).

Concerning the strictness of the inclusion, it will be a consequence of Propositions 8 and 10. □

Proposition 10. *GTT $\not\subseteq$ Rec$_\times$ and Rec$_\times$ $\not\subseteq$ GTT.*

Proof. Δ is accepted by a GTT (with no state and no transition) but it does not belong to *Rec*$_\times$. On the other hand, if $\mathcal{F} = \{f, a\}$, then $\{a, f(a)\}^2$ is in *Rec*$_\times$ (it is the product of two finite languages) but it is not accepted by any GTT since any GTT accepts at least Δ. □

Finally, there is an example of a relation R_c which is in *Rec* and not in the union *Rec*$_\times \cup$ GTT; consider for instance the alphabet $\{a(), b(), 0\}$ and the one step reduction relation associated with the rewrite system $a(x) \rightarrow x$. In other words,

$$(u,v) \in R_c \Longleftrightarrow \exists C \in \mathcal{C}(\mathcal{F}), \exists t \in T(\mathcal{F}), u = C[a(t)] \wedge v = C[t]$$

It is left as an exercise to prove that $R_c \in$ *Rec* $\setminus$ (*Rec*$_\times \cup$ GTT).

3.2.4 Closure properties for *Rec*$_\times$ and *Rec*; cylindrification and projection

Let us start with the classical closure properties.

Proposition 11. *Rec$_\times$ and Rec are closed under Boolean operations.*

The proof of this proposition is straightforward and left as an exercise.

These relations are also closed under *cylindrification* and *projection*. Let us first define these operations which are specific to automata on tuples:

Definition 6. *If $R \subseteq T(\mathcal{F})^n$ $(n \geq 1)$ and $1 \leq i \leq n$ then the i*th projection *of R is the relation $R_i \subseteq T(\mathcal{F})^{n-1}$ defined by*

$$R_i(t_1, \ldots, t_{n-1}) \Leftrightarrow \exists t \in T(\mathcal{F}) \;\; R(t_1, \ldots, t_{i-1}, t, t_i, \ldots, t_{n-1})$$

When $n = 1$, $T(\mathcal{F})^{n-1}$ is by convention a singleton set $\{\top\}$ (so as to keep the property that $T(\mathcal{F})^{n+1} = T(\mathcal{F}) \times T(\mathcal{F})^n$). $\{\top\}$ is assumed to be a neutral element w.r.t. Cartesian product. In such a situation, a relation $R \subseteq T(\mathcal{F})^0$ is either $\emptyset$ or $\{\top\}$ (it is a propositional variable).

Definition 7. *If $R \subseteq T(\mathcal{F})^n$ $(n \geq 0)$ and $1 \leq i \leq n+1$, then the i*th cylindrification *of R is the relation $R_i \subseteq T(\mathcal{F})^{n+1}$ defined by*

$$R_i(t_1, \ldots, t_{i-1}, t, t_i, \ldots, t_n) \Leftrightarrow R(t_1, \ldots, t_{i-1}, t_i, \ldots, t_n)$$

Proposition 12. *$Rec_\times$ and Rec are effectively closed under projection and cylindrification. Actually, ith projection can be computed in linear time and the ith cylindrification of $\mathcal{A}$ can be computed in linear time (assuming that the size of the alphabet is constant).*

Proof. For $Rec_\times$, this property is easy: projection on the ith component simply amounts to remove the ith automaton. Cylindrification on the ith component simply amounts to insert as a ith automaton, an automaton accepting all terms.

Assume that $R \in Rec$. The ith projection of R is simply its image by the following linear tree homomorphism:

$$h_i([f_1, \ldots, f_n](t_1, \ldots, t_k)) \stackrel{\text{def}}{=} [f_1 \ldots f_{i-1} f_{i+1} \ldots f_n](h_i(t_1), \ldots, h_i(t_m))$$

in which m is the arity of $[f_1 \ldots f_{i-1} f_{i+1} \ldots f_n]$ (which is smaller or equal to k). Hence, by Theorem 6, the ith projection of R is recognizable (and we can extract from the proof a linear construction of the automaton).

Similarly, the ith cylindrification is obtained as an inverse homomorphic image, hence is recognizable thanks to Theorem 7.

□

Note that using the above construction, the projection of a deterministic automaton may be non-deterministic (see exercises)

Example 29. Let $\mathcal{F} = \{f, g, a\}$ where f is binary, g is unary and a is a constant. Consider the following automaton $\mathcal{A}$ on $\mathcal{F}' = (\mathcal{F} \cup \{\perp\})^2$: The set of states is $\{q_1, q_2, q_3, q_4, q_5\}$ and the set of final states is $\{q_3\}$[1]

$$\begin{array}{rcl@{\qquad}rcl}
a\perp & \rightarrow & q_1 & f\perp(q_1, q_1) & \rightarrow & q_1 \\
g\perp(q_1) & \rightarrow & q_1 & fg(q_2, q_1) & \rightarrow & q_3 \\
ga(q_1) & \rightarrow & q_2 & f\perp(q_4, q_1) & \rightarrow & q_4 \\
g\perp(q_1) & \rightarrow & q_4 & fa(q_4, q_1) & \rightarrow & q_2 \\
gg(q_3) & \rightarrow & q_3 & ff(q_3, q_3) & \rightarrow & q_3 \\
aa & \rightarrow & q_5 & ff(q_3, q_5) & \rightarrow & q_3 \\
gg(q_5) & \rightarrow & q_5 & ff(q_5, q_3) & \rightarrow & q_3 \\
ff(q_5, q_5) & \rightarrow & q_5 & & &
\end{array}$$

[1] This automaton accepts the set of pairs of terms (u, v) such that u can be rewritten in one or more steps to v by the rewrite system $f(g(x), y) \rightarrow g(a)$.

The first projection of this automaton gives:

$$\begin{array}{rclrcl}
a & \rightarrow & q_2 & g(q_3) & \rightarrow & q_3 \\
a & \rightarrow & q_5 & g(q_5) & \rightarrow & q_5 \\
g(q_2) & \rightarrow & q_3 & f(q_3,q_3) & \rightarrow & q_3 \\
f(q_3,q_5) & \rightarrow & q_3 & f(q_5,q_5) & \rightarrow & q_5 \\
f(q_5,q_3) & \rightarrow & q_3 & & &
\end{array}$$

Which accepts the terms containing $g(a)$ as a subterm[2].

3.2.5 Closure of GTT by composition and iteration

Theorem 20. *If $R \subseteq T(\mathcal{F})^2$ is recognized by a GTT, then its transitive closure R^* is also recognized by a GTT.*

The detailed proof is technical, so let us show it on a picture and explain it informally.

We consider two terms (t, v) and (v, u) which are both accepted by the GTT and we wish that (t, u) is also accepted. For simplicity, consider only one state q such that $t \xrightarrow[\mathcal{A}_1]{*} C[q] \xleftarrow[\mathcal{A}_2]{*} v$ and $v \xrightarrow[\mathcal{A}_1]{*} C'[q_1, \dots, q_n] \xleftarrow[\mathcal{A}_2]{*} u$. There are actually two cases: C can be "bigger" than C' or "smaller". Assume it is smaller. Then q is reached at a position inside C': $C' = C[C'']_p$. The situation is depicted on Figure 3.5. Along the reduction of v to q by $\mathcal{A}_2$, we enter a configuration $C''[q'_1, \dots, q'_n]$. The idea now is to add to $\mathcal{A}_2$ ϵ-transitions from q_i to q'_i. In this way, as can easily be seen on Figure 3.5, we get a reduction from u to $C[q]$, hence the pair (t, u) is accepted.

Proof. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the pair of automata defining the GTT which accepts R. We compute by induction the automata $\mathcal{A}_1^n, \mathcal{A}_2^n$. $\mathcal{A}_i^0 = \mathcal{A}_i$ and $\mathcal{A}_i^{n+1}$ is obtained by adding new transitions to $\mathcal{A}_i^n$: Let Q_i be the set of states of $\mathcal{A}_i$ (and also the set of states of $\mathcal{A}_i^n$).

- If $L_{\mathcal{A}_2^n}(q) \cap L_{\mathcal{A}_1^n}(q') \neq \emptyset$, $q \in Q_1 \cap Q_2$ and $q \not\xrightarrow[\mathcal{A}_1^n]{*} q'$, then $\mathcal{A}_1^{n+1}$ is obtained from $\mathcal{A}_1^n$ by adding the ϵ-transition $q \rightarrow q'$ and $\mathcal{A}_2^{n+1} = \mathcal{A}_2^n$.
- If $L_{\mathcal{A}_1^n}(q) \cap L_{\mathcal{A}_2^n}(q') \neq \emptyset$, $q \in Q_1 \cap Q_2$ and $q \not\xrightarrow[\mathcal{A}_2^n]{*} q'$, then $\mathcal{A}_2^{n+1}$ is obtained from $\mathcal{A}_2^n$ by adding the ϵ-transition $q \rightarrow q'$ and $\mathcal{A}_1^{n+1} = \mathcal{A}_1^n$.

If there are several ways of obtaining $\mathcal{A}_i^{n+1}$ from $\mathcal{A}_i^n$ using these rules, we don't care which of these ways is used.

First, these completion rules are decidable by the decision properties of chapter 1. Their application also terminates as at each application strictly decreases $k_1(n) + k_2(n)$ where $k_i(n)$ is the number of pairs of states $(q, q') \in Q_1 \cup Q_2$ such that there is no ϵ-transition in $\mathcal{A}_i^n$ from q to q'. We let $\mathcal{A}_i^*$ be a fixed point of this computation. We show that $(\mathcal{A}_1^*, \mathcal{A}_2^*)$ defines a GTT accepting R^*.

[2]i.e. the terms that are obtained by applying at least one rewriting step using $f(g(x), y) \rightarrow g(a)$

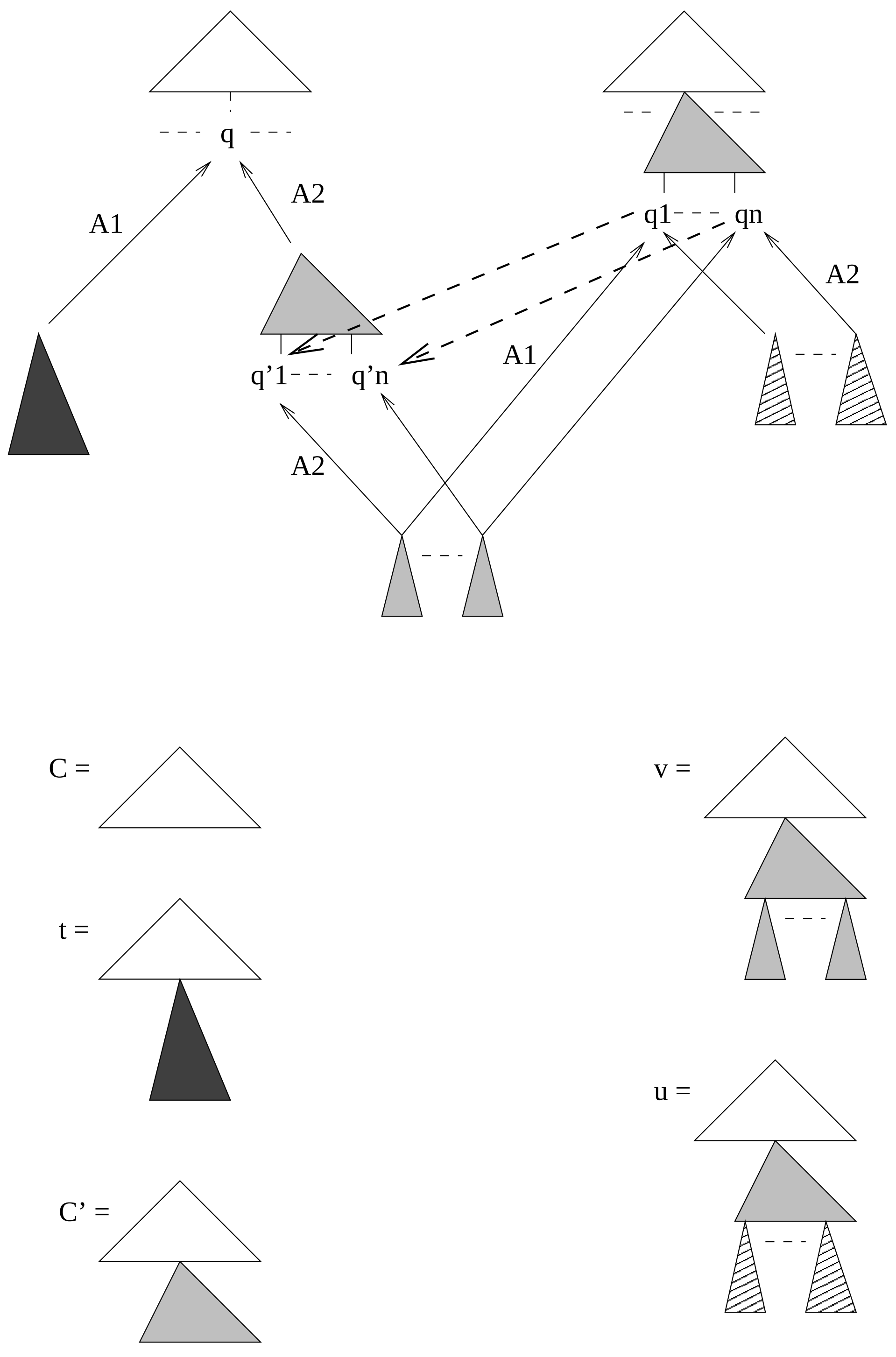


Figure 3.5: The proof of Theorem 20

- Each pair of terms accepted by the GTT $(\mathcal{A}_1^*, \mathcal{A}_2^*)$ is in R^*: we show by induction on n that each pair of terms accepted by $(\mathcal{A}_1^n, \mathcal{A}_2^n)$ is in R^*. For $n = 0$, this follows from the hypothesis. Let us now assume that $\mathcal{A}_1^{n+1}$ is obtained by adding $q \rightarrow q'$ to the transitions of $\mathcal{A}_1^n$ (The other case is symmetric). Let (t, u) be accepted by the GTT $(\mathcal{A}_1^{n+1}, \mathcal{A}_2^{n+1})$. By definition, there is a context C and positions $p_1, \ldots, p_k$ such that $t = C[t_1, \ldots, t_k]_{p_1,\ldots,p_k}$, $u = C[u_1, \ldots, u_k]_{p_1,\ldots,p_k}$ and there are states $q_1, \ldots, q_k \in Q_1 \cap Q_2$ such that, for all i, $t_i \xrightarrow[\mathcal{A}_1^{n+1}]{*} q_i$ and $u_i \xrightarrow[\mathcal{A}_2^n]{*} q_i$.

  We prove the result by induction on the number m of times $q \rightarrow q'$ is applied in the reductions $t_i \xrightarrow[\mathcal{A}_1^{n+1}]{*} q_i$. If $m = 0$. Then this is the first induction hypothesis: (t, u) is accepted by $(\mathcal{A}_1^n, \mathcal{A}_2^n)$, hence $(t, u) \in R^*$. Now, assume that, for some i,

  $$t_i \xrightarrow[\mathcal{A}_1^{n+1}]{*} t_i'[q]_p \xrightarrow[q \rightarrow q']{*} t_i'[q']_p \xrightarrow[\mathcal{A}_1^n]{*} q_i$$

  By definition, there is a term v such that $v \xrightarrow[\mathcal{A}_2^n]{*} q$ and $v \xrightarrow[\mathcal{A}_1^n]{*} q'$. Hence

  $$t_i[v]_p \xrightarrow[\mathcal{A}_1^{n+1}]{*} q_i$$

  And the number of reduction steps using $q \rightarrow q'$ is strictly smaller here than in the reduction from t_i to q_i. Hence, by induction hypothesis, $(t[v]_{p_ip}, u) \in R^*$. On the other hand, $(t, t[v]_{p_ip})$ is accepted by the GTT $(\mathcal{A}_1^{n+1}, \mathcal{A}_2^n)$ since $t|_{p_ip} \xrightarrow[\mathcal{A}_1^{n+1}]{*} q$ and $v \xrightarrow[\mathcal{A}_2^n]{*} q$. Moreover, by construction, the first sequence of reductions uses strictly less than m times the transition $q \rightarrow q'$. Then, by induction hypothesis, $(t, t[v]_{p_ip}) \in R^*$. Now from $(t, t[v]_{p_ip} \in R^*$ and $(t[v]_{p_ip}, u) \in R^*$, we conclude $(t, u) \in R^*$.

- If $(t, u) \in R^*$, then (t, u) is accepted by the GTT $(\mathcal{A}_1^*, \mathcal{A}_2^*)$. Let us prove the following intermediate result:

  Lemma 1. *if* $t \xrightarrow[\mathcal{A}_1^*]{*} q$, $v \xrightarrow[\mathcal{A}_2^*]{*} q$, $v \xrightarrow[\mathcal{A}_1^*]{*} C[q_1, \ldots, q_n]_{p_1,\ldots,p_n}$ *and* $u \xrightarrow[\mathcal{A}_2^*]{*} C[q_1, \ldots, q_n]_{p_1,\ldots,p_n}$, *then* $u \xrightarrow[\mathcal{A}_2^*]{*} q$ *(and hence* (t, u) *is accepted by the GTT)*

  Let $v \xrightarrow[\mathcal{A}_2^*]{*} C[q_1', \ldots, q_n']_{p_1,\ldots,p_n} \xrightarrow[\mathcal{A}_2^*]{*} q$. For each i, $v|_{p_i} \in L_{\mathcal{A}_2^*}(q_i') \cap L_{\mathcal{A}_1^*}(q_i)$ and $q_i \in Q_1 \cap Q_2$. Hence, by construction, $q_i \xrightarrow[\mathcal{A}_2^*]{} q_i'$. It follows that

  $$u \xrightarrow[\mathcal{A}_2^*]{*} C[q_1, \ldots q_n]_{p_1,\ldots,p_n} \xrightarrow[\mathcal{A}_2^*]{*} C[q_1', \ldots, q_n']_{p_1,\ldots,p_n} \xrightarrow[\mathcal{A}_2^*]{*} q$$

  Which proves our lemma.

  Symmetrically, if $t \xrightarrow[\mathcal{A}_1^*]{*} C[q_1, \ldots, q_n]_{p_1,\ldots,p_n}$, $v \xrightarrow[\mathcal{A}_2^*]{*} C[q_1, \ldots, q_n]_{p_1,\ldots,p_n}$, $v \xrightarrow[\mathcal{A}_1^*]{*} q$ and $u \xrightarrow[\mathcal{A}_2^*]{*} q$, then $t \xrightarrow[\mathcal{A}_1^*]{*} q$

  Now, let $(t, u) \in R^n$: we prove that (t, u) is accepted by the GTT $(\mathcal{A}_1^*, \mathcal{A}_2^*)$ by induction on n. If $n = 1$, then the result follows from the inclusion of $L(\mathcal{A}_1, \mathcal{A}_2)$ in $L(\mathcal{A}_1^*, \mathcal{A}_2^*)$. Now, let v be such that $(t, v) \in R$ and

$(v, u) \in R^{n-1}$. By induction hypothesis, both (t, v) and (v, u) are accepted by the GTT $(\mathcal{A}_1^*, \mathcal{A}_2^*)$: there are context C and C' and positions $p_1, \ldots, p_k, p'_1, \ldots, p'_m$ such that $t = C[t_1, \ldots, t_k]_{p_1,\ldots,p_k}$, $v = C[v_1, \ldots, v_k]_{p_1,\ldots,p_k}$, $v = C'[v'_1, \ldots, v'_m]_{p'_1,\ldots,p'_m}$, $u = C'[u_1, \ldots, u_m]$ and states $q_1, \ldots, q_k, q'_1, \ldots, q'_m \in Q_1 \cap Q_2$ such that for all i, j, $t_i \xrightarrow[\mathcal{A}_1^*]{*} q_i$, $v_i \xrightarrow[\mathcal{A}_2^*]{*} q_i$, $v'_j \xrightarrow[\mathcal{A}_1^*]{*} q'_j$, $u_j \xrightarrow[\mathcal{A}_2^*]{*} q'_j$.
Let C'' be the largest context more general than C and C'; the positions of C'' are the positions of both $C[q_1, \ldots, q_n]_{p_1,\ldots,p_n}$ and $C'[q'_1, \ldots, q'_m]_{p'_1,\ldots,p'_m}$.
C'', $p''_1, \ldots, p''_l$ are such that:

- For each $1 \leq i \leq l$, there is a j such that either $p_j = p''_i$ or $p'_j = p''_i$
- For all $1 \leq i \leq n$ there is a j such that $p_i \geq p''_j$
- For all $1 \leq i \leq m$ there is a j such that $p'_i \geq p''_j$
- the positions p''_j are pairwise incomparable w.r.t. the prefix ordering.

Let us fix a $j \in [1..l]$. Assume that $p''_j = p_i$ (the other case is symmetric). We can apply our lemma to $t_j = t|_{p''_j}$ (in place of t), $v_j = v|_{p''_j}$ (in place of v) and $u|_{p''_j}$ (in place of u), showing that $u|_{p''_j} \xrightarrow[\mathcal{A}_2^*]{*} q_i$. If we let now $q''_j = q_i$ when $p''_j = p_i$ and $q''_j = q'_i$ when $p''_j = p'_i$, we get

$$t \xrightarrow[\mathcal{A}_1^*]{*} C''[q''_1, \ldots, q''_l]_{p''_1,\ldots,p''_l} \xleftarrow[\mathcal{A}_2^*]{*} u$$

which completes the proof.

□

Proposition 13. *If R and R' are in GTT then their composition $R \circ R'$ is also in GTT.*

Proof. Let $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{A}'_1, \mathcal{A}'_2)$ be the two pairs of automata which recognize R and R' respectively. The construction of the GTT $(\mathcal{A}_1^*, \mathcal{A}_2^*)$ accepting $R \circ R'$ is similar to the proof of Theorem 20: we construct a sequence of automata $\mathcal{A}_1^n, \mathcal{A}_2^n$ by induction on n, with $\mathcal{A}_1^0 = \mathcal{A}_1$ and $\mathcal{A}_2^0 = \mathcal{A}'_2$ and

- If there is a state q of $\mathcal{A}_2$ and a state q' of both $\mathcal{A}'_1$ and $\mathcal{A}_2^n$ such that there is a term t accepted in both q by $\mathcal{A}_2$ and q' by $\mathcal{A}'_1$, then we add the transition $q' \to q$ to $\mathcal{A}_2^n$ and $\mathcal{A}_1^n$ remains unchanged

- If there is a state q of $\mathcal{A}'_1$ and a state q' of both $\mathcal{A}_1^n$ and $\mathcal{A}_2$ such that there is a term t accepted in both state q' by $\mathcal{A}_2$ and in state q by $\mathcal{A}'_1$, then we add the transition $q' \to q$ to $\mathcal{A}_1^n$ and $\mathcal{A}_2^n$ remains unchanged.

Figure 3.5 also illustrates this construction, with slight changes.
The proof goes on like in Theorem 20 with slight changes. (Actually, it is a bit simpler here). □

GTTs do not have many other good closure properties (see the exercises).

3.3 The logic WSkS

3.3.1 Syntax

Terms of WSkS are formed out of the constant ϵ, first-order variable symbols (typically written with lower-case letters $x, y, z, x', x_1, ...$) and unary symbols $1, \ldots, n$ written in postfix notation. For instance $x1123, \epsilon 2111$ are terms. The latter will be often written omitting ϵ (e.g. 2111 instead of $\epsilon 2111$).

Atomic formulas are either equalities $s = t$ between terms, inequalities $s \leq t$ or $s \geq t$ between terms, or membership constraints $t \in X$ where t is a term and X is a *second-order variable symbol.* Second-order variables will be typically denoted using upper-case letters.

Formulas are built from the atomic formulas using the logical connectives $\wedge, \vee, \neg, \Rightarrow, \Leftarrow, \Leftrightarrow$ and the quantifiers $\exists x, \forall x$(quantification on individuals)$\exists X, \forall X$ (quantification on sets); we may quantify both first-order and second-order variables.

As usual, we do not need all this artillery: we may stick to a subset of logical connectives (and even a subset of atomic formulas as will be discussed in Section 3.3.4). For instance $\phi \Leftrightarrow \psi$ is an abbreviation for $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$, $\phi \Rightarrow \psi$ is another way of writing $\psi \Leftarrow \phi$, $\phi \Rightarrow \psi$ is an abbreviation for $(\neg\phi) \vee \psi$, $\forall x.\phi$ stands for $\neg\exists x.\neg\phi$ etc ... We will use the extended syntax for convenience, but we will restrict ourselves to the atomic formulas $s = t, s \leq t, t \in X$ and the logical connectives $\vee, \neg, \exists x, \exists X$ in the proofs.

The set of *free variables* of a formula ϕ is defined as usual.

3.3.2 Semantics

We consider the particular interpretation where terms are strings belonging to $\{1, \ldots, k\}^*$, $=$ is the equality of strings, and $\leq$ is interpreted as the prefix ordering. Second order variables are interpreted as *finite* subsets of $\{1, \ldots, k\}^*$, so $\in$ is then the membership predicate.

Let $t_1, \ldots, t_n \in \{1, \ldots, k\}^*$ and $S_1, \ldots, S_n$ be finite subsets of $\{1, \ldots, k\}^*$. Given a formula $\phi(x_1, \ldots, x_n, X_1, \ldots, X_m)$ with free variables $x_1, \ldots, x_n, X_1, \ldots, X_m$, the assignment $\{x_1 \mapsto t_1, \ldots x_n \mapsto t_n, X_1 \mapsto S_1, \ldots X_m \mapsto S_m\}$ *satisfies* ϕ, which is written $\sigma \models \phi$ (or also $t_1, \ldots, t_n, S_1, \ldots, S_m \models \phi$) if replacing the variables with their corresponding value, the formula holds in the above model.

Remark: the logic SkS is defined as above, except that set variables may be interpreted as infinite sets.

3.3.3 Examples

We list below a number of formulas defining predicates on sets and singletons. After these examples, we may use the below-defined abbreviations as if there were primitives of the logic.

X is a subset of Y:

$$X \subseteq Y \stackrel{\text{def}}{=} \forall x.(x \in X \Rightarrow x \in Y)$$

Finite union:

$$X = \bigcup_{i=1}^{n} X_i \stackrel{\text{def}}{=} \bigwedge_{i=1}^{n} X_i \subseteq X \wedge \forall x.(x \in X \Rightarrow \bigvee_{i=1}^{n} x \in X_n)$$

Intersection:

$$X \cap Y = Z \stackrel{\text{def}}{=} \forall x.x \in Z \Leftrightarrow (x \in X \wedge x \in Y)$$

Partition:

$$\text{Partition}(X, X_1, \ldots, X_n) \stackrel{\text{def}}{=} X = \bigcup_{i=1}^{n} X_i \wedge \bigwedge_{i=1}^{n-1} \bigwedge_{j=i+1}^{n} X_i \cap X_j = \emptyset$$

X is closed under prefix:

$$\text{PrefixClosed}(X) \stackrel{\text{def}}{=} \forall z.\forall y.(z \in X \wedge y \leq z) \Rightarrow y \in X$$

Set equality:

$$Y = X \stackrel{\text{def}}{=} Y \subseteq X \wedge X \subseteq Y$$

Emptiness:

$$X = \emptyset \stackrel{\text{def}}{=} \forall Y.(Y \subseteq X \Rightarrow Y = X)$$

X is a Singleton:

$$\text{Sing}(X) \stackrel{\text{def}}{=} X \neq \emptyset \wedge \forall Y(Y \subseteq X \Rightarrow (Y = X \vee Y = \emptyset)$$

The prefix ordering:

$$x \leq y \stackrel{\text{def}}{=} \forall X.(x \in X \wedge (\forall z.((z \in X \wedge z \neq y) \Rightarrow \bigwedge_{i=1}^{k} zi \in X))) \Rightarrow y \in X$$

"every set containing x and closed by successors contains y"

This shows that $\leq$ can be removed from the syntax of WSkS formulas without decreasing the expressive power of the logic.

Coding of trees: assume that k is the maximal arity of a function symbol in $\mathcal{F}$. If $t \in T(\mathcal{F})$ $C(t)$ is the tuples of sets $(S, S_{f_1}, \ldots, S_{f_n})$ if $\mathcal{F} = \{f_1, \ldots, f_n\}$, $S = \bigcup_{i=1}^{n} S_{f_i}$ and S_{f_i} is the set of positions in t which are labeled with f_i.

For instance $C(f(g(a), f(a, b)))$ is the tuple $S = \{\Lambda, 1, 11, 2, 21, 22\}$, $S_f = \{\Lambda, 2\}$, $S_g = \{1\}$, $S_a = \{11, 21\}$, $S_b = \{22\}$.

$(S, S_{f_1}, \ldots, S_{f_n})$ is the coding of some $t \in T(\mathcal{F})$ is defined by:

$$\begin{array}{rl} \text{Term}(X, X_1, \ldots, X_n) \stackrel{\text{def}}{=} & \text{Partition}(X, X_1, \ldots, X_n) \quad \wedge \text{PrefixClosed}(X) \\ \wedge & \bigwedge_{i=1}^{k} \bigwedge_{a(f_j)=i} (\bigwedge_{l=1}^{i} \forall x.(x \in X_{f_j} \Rightarrow xl \in X) \\ & \qquad \wedge \bigwedge_{l=i+1}^{k} \forall y.(y \in X_{f_j} \Rightarrow xl \notin X)) \end{array}$$

3.3.4 Restricting the syntax

If we consider that a first-order variable is a singleton set, it is possible to transform any formula into an equivalent one which does not contain any first-order variable.

More precisely, we consider now that formulas are built upon the atomic formulas:

$$X \subseteq Y, \mathrm{Sing}(X), X = Yi, X = \epsilon$$

using the logical connectives and second-order quantification only. Let us call this new syntax the *restricted syntax.*

These formulas are interpreted as expected. In particular $\mathrm{Sing}(X)$ holds true when X is a singleton set and $X = Yi$ holds true when X and Y are singleton sets $\{s\}$ and $\{t\}$ respectively and $s = ti$. Let us write $\models_2$ the satisfaction relation for this new logic.

Proposition 14. *There is a translation T from WSkS formulas to the restricted syntax such that*

$$s_1, \ldots, s_n, S_1, \ldots, S_m \models \phi(x_1, \ldots, x_n, X_1, \ldots, X_m)$$

if and only if

$$\{s_1\}, \ldots, \{s_n\}, S_1, \ldots, S_m \models_2 T(\phi)(X_{x_1}, \ldots, X_{x_n}, X_1, \ldots, X_m)$$

Conversely, there is a translation T' such from the restricted syntax to WSkS such that

$$S_1, \ldots, S_m \models T'(\phi)(X_1, \ldots, X_m)$$

if and only if

$$S_1, \ldots, S_m \models_2 \phi(X_1, \ldots, X_m))$$

Proof. First, according to the previous section, we can restrict our attention to formulas built upon the only atomic formulas $t \in X$ and $s = t$. Then, each atomic formula is flattened according to the rules:

$$\begin{array}{rcl} ti \in X & \rightarrow & \exists y. y = ti \wedge y \in X \\ xi = yj & \rightarrow & \exists z. z = xi \wedge z = yj \\ ti = s & \rightarrow & \exists z. z = t \wedge zi = s \end{array}$$

The last rule assumes that t is not a variable

Next, we associate a second-order variable X_y to each first-order variable y and transform the flat atomic formulas:

$$\begin{array}{rcl} T(y \in X) & \stackrel{\text{def}}{=} & X_y \subseteq X \\ T(y = xi) & \stackrel{\text{def}}{=} & X_y = X_x i \\ T(x = \epsilon) & \stackrel{\text{def}}{=} & X_x = \epsilon \end{array}$$

The translation of other flat atomic formulas can be derived from these ones.

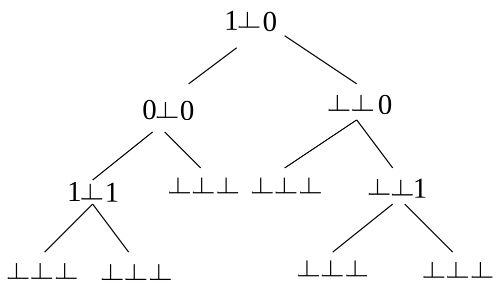


Figure 3.6: An example of a tree coding a triple of finite sets of strings

Now, $T(\phi \vee \psi) \stackrel{\text{def}}{=} T(\phi) \vee T(\psi)$, $T(\neg(\phi)) \stackrel{\text{def}}{=} \neg T(\phi)$, $T(\exists X.\phi) \stackrel{\text{def}}{=} \exists X.T(\phi)$, $T(\exists y.\phi) \stackrel{\text{def}}{=} \exists X_y.\text{Sing}(X_y) \wedge T(\phi)$. Finally, we add $\text{Sing}(X_x)$ for each free variable x.

For the converse, the translation T' has been given in the previous section, except for the atomic formulas $X = Yi$ (which becomes $\text{Sing}(X) \wedge \text{Sing}(Y) \wedge \exists x \exists y. x \in X \wedge y \in Y \wedge x = yi$) and $X = \epsilon$ (which becomes $\text{Sing}(X) \wedge \forall x. x \in X \Rightarrow x = \epsilon$).

$\square$

3.3.5 Definable sets are recognizable sets

Definition 8. *A set L of tuples of finite sets of words is* definable in WSkS *if there is a formula ϕ of WSkS with free variables $X_1, \ldots, X_n$ such that $(S_1, \ldots, S_n) \in L$ if and only if $S_1, \ldots, S_n \models \phi$.*

Each tuple of finite sets of words $S_1, \ldots, S_n \subseteq \{1, \ldots, k\}^*$ is identified to a finite tree $(S_1, \ldots, S_n)^\sim$ over the alphabet $\{0, 1, \perp\}^n$ where any string containing a 0 or a 1 is k-ary and $\perp^n$ is a constant symbol, in the following way[3]:

$$\mathcal{P}os((S_1, \ldots, S_n)^\sim) \stackrel{\text{def}}{=} \{\epsilon\} \cup \{pi \mid \exists p' \in \bigcup_{i=1}^{n} S_i, p \leq p', i \in \{1, \ldots, k\}\}$$

is the set of prefixes of words in some S_i. The symbol at position p: $(S_1, \ldots, S_n)^\sim(p) = \alpha_1 \ldots \alpha_n$ is defined as follows:

- $\alpha_i = 1$ if and only if $p \in S_i$
- $\alpha_i = 0$ if and only if $p \notin S_i$ and $\exists p' \in S_i$ and $\exists p''. p \cdot p'' = p'$
- $\alpha_i = \perp$ otherwise.

Example 30. Consider for instance $S_1 = \{\epsilon, 11\}$, $S_2 = \emptyset$, $S_3 = \{11, 22\}$ three subsets of $\{1, 2\}^*$. Then the coding $(S_1, S_2, S_3)^\sim$ is depicted on Figure 3.6.

[3]This is very similar to the coding of Section 3.2.1

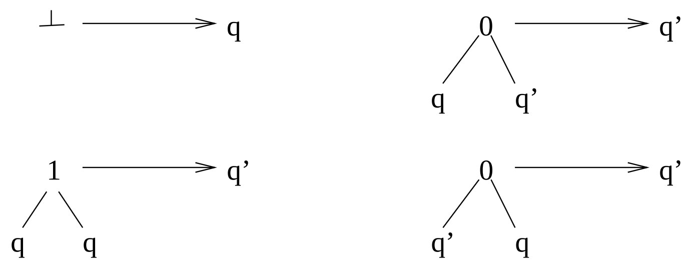


Figure 3.7: The automaton for Sing(X)

Lemma 2. *If a set L of tuples of finite subsets of $\{1, \dots, k\}^*$ is definable in WSkS, then $\widetilde{L} \overset{def}{=} \{(S_1, \dots, S_n)^\sim \mid (S_1, \dots, S_n) \in L\}$ is in Rec.*

Proof. By Proposition 14, if L is definable in WSkS, it is also definable with the restricted syntax. We are going now to prove the lemma by induction on the structure of the formula ϕ which defines L. We assume that all variables in ϕ are bound at most once in the formula and we also assume a fixed total ordering $\leq$ on the variables. If ψ is a subformula of ϕ with free variables $Y_1 < \ldots < Y_n$, we construct an automaton $\mathcal{A}_\psi$ working on the alphabet $\{0, 1, \perp\}^n$ such that $(S_1, \dots, S_n) \models_2 \psi$ if and only if $(S_1, \dots, S_n)^\sim \in L(\mathcal{A}_\psi)$

The base case consists in constructing an automaton for each atomic formula. (We assume here that $k = 2$ for simplicity, but this works of course for arbitrary k).

The automaton $\mathcal{A}_{\text{Sing}(X)}$ is depicted on Figure 3.7. The only final state is q'.

The automaton $\mathcal{A}_{X \subseteq Y}$ (with $X < Y$) is depicted on Figure 3.8. The only state (which is also final) is q.

The automaton $\mathcal{A}_{X=Y1}$ is depicted on Figure 3.9. The only final state is q''. An automaton for $X = Y2$ is obtained in a similar way.

The automaton for $X = \epsilon$ is depicted on Figure 3.10 (the final state is q').

Now, for the induction step, we have several cases to investigate:

- If ϕ is a disjunction $\phi_1 \vee \phi_2$, where $\vec{X_i}$ are the set of free variables of ϕ_i respectively. Then we first cylindrify the automata for ϕ_1 and ϕ_2 respectively in such a way that they recognize the solutions of ϕ_1 and ϕ_2, with free variables $\vec{X_1} \cup \vec{X_2}$. (See Proposition 12).More precisely, let $\vec{X_1} \cup \vec{X_2} = \{Y_1, \dots, Y_n\}$ with $Y_1 < \ldots < Y_n$. Then we successively apply the ith cylindrification to the automaton of ϕ_1 (resp. ϕ_2) for the variables Y_i which are not free in ϕ_1 (resp. ϕ_2). Then the automaton $\mathcal{A}_\phi$ is obtained as the union of these automata. (*Rec* is closed under union by Proposition 11).
- If ϕ is a formula $\neg\phi_1$ then $\mathcal{A}_\phi$ is the automaton accepting the complement of $\mathcal{A}_{\phi_1}$. (See Theorem 5)
- If ϕ is a formula $\exists X.\phi_1$. Assume that X correspond to the ith component. Then $\mathcal{A}_\phi$ is the ith projection of $\mathcal{A}_{\phi_1}$ (see Proposition 12).

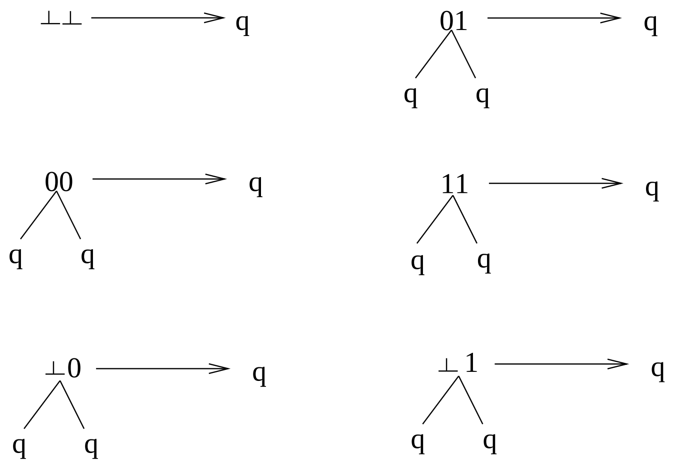


Figure 3.8: The automaton for $X \subseteq Y$

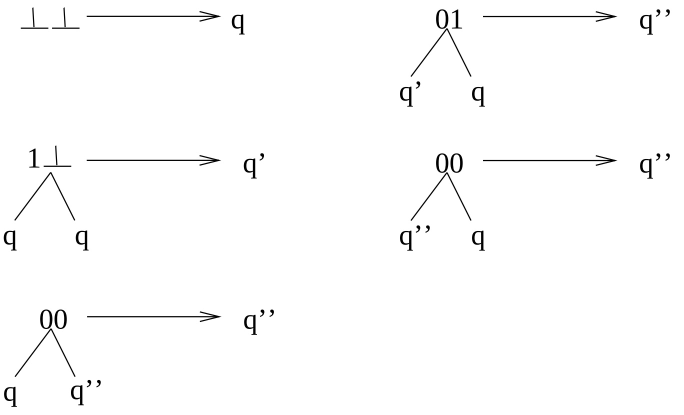


Figure 3.9: The automaton for $X = Y1$

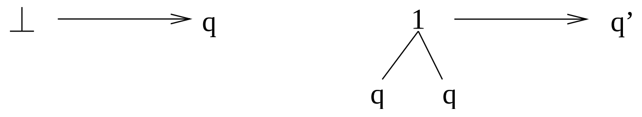


Figure 3.10: The automaton for $X = \epsilon$

$\square$

Example 31. Consider the following formula, with free variables X, Y:

$$\forall x, y.(x \in X \wedge y \in Y) \Rightarrow \neg(x \geq y)$$

We want to compute an automaton which accepts the assignments to X, Y satisfying the formula. First, write the formula as

$$\neg\exists X_1, Y_1. \wedge X_1 \subseteq X \wedge Y_1 \subseteq Y \wedge G(X_1, Y_1)$$

where $G(X_1, Y_1)$ expresses that X_1 is a singleton x, Y_1 is a singleton y and $x \geq y$. We can use the definition of $\geq$ as a WS2S formula, or compute directly the automaton, yielding

$$\begin{array}{rcl@{\qquad}rcl}
\perp\perp & \rightarrow & q & 11(q,q) & \rightarrow & q_2 \\
1\perp(q,q) & \rightarrow & q_1 & 0\perp(q,q_1) & \rightarrow & q_1 \\
0\perp(q_1,q) & \rightarrow & q_1 & 01(q_1,q) & \rightarrow & q_2 \\
01(q,q_2) & \rightarrow & q_2 & 00(q_2,q) & \rightarrow & q_2 \\
00(q,q_2) & \rightarrow & q_2 & & &
\end{array}$$

where q_2 is the only final state. Now, using cylindrification, intersection, projection and negation we get the following automaton (intermediate steps yield large automata which would require a full page to be displayed):

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\perp\perp & \rightarrow & q_0 & \perp 1(q_0,q_0) & \rightarrow & q_1 & 1\perp(q_0,q_0) & \rightarrow & q_2 \\
\perp 0(q_0,q_1) & \rightarrow & q_1 & \perp 0(q_1,q_0) & \rightarrow & q_1 & \perp 0(q_1,q_1) & \rightarrow & q_1 \\
0\perp(q_0,q_2) & \rightarrow & q_2 & 0\perp(q_2,q_0) & \rightarrow & q_2 & 0\perp(q_2,q_2) & \rightarrow & q_2 \\
\perp 1(q_0,q_1) & \rightarrow & q_1 & \perp 1(q_1,q_0) & \rightarrow & q_1 & \perp 1(q_1,q_1) & \rightarrow & q_1 \\
1\perp(q_0,q_2) & \rightarrow & q_2 & 1\perp(q_2,q_0) & \rightarrow & q_2 & 1\perp(q_2,q_2) & \rightarrow & q_2 \\
10(q_1,q_0) & \rightarrow & q_3 & 10(q_0,q_1) & \rightarrow & q_3 & 10(q_1,q_1) & \rightarrow & q_3 \\
10(q_1,q_2) & \rightarrow & q_3 & 10(q_2,q_1) & \rightarrow & q_3 & 10(q_i,q_3) & \rightarrow & q_3 \\
10(q_3,q_i) & \rightarrow & q_3 & 00(q_i,q_3) & \rightarrow & q_3 & 00(q_3,q_i) & \rightarrow & q_3
\end{array}$$

where i ranges over $\{0, 1, 2, 3\}$ and q_3 is the only final state.

3.3.6 Recognizable sets are definable

As in previous section, we use a representation of (tuples of) terms as tuples of set variables; if $(t_1, \ldots, t_n) \in T(\mathcal{F})^n$, we write $\overline{[t_1, \ldots, t_n]}$ the $(|\mathcal{F}| + 1)^n + 1$-tuple of finite sets which represents it: one set for the positions of $[t_1, \ldots, t_n]$ and one set for each element of the alphabet $(\mathcal{F} \cup \{\perp\})^n$. As it has been seen in previous section, there is a WSkS formula Term($[t_1, \ldots, t_n]$) which expresses the image of the coding.

Lemma 3. *Every relation in Rec is definable. More precisely, if $R \in Rec$ there is a formula ϕ such that $(S_1, \ldots, S_m) \models_2 \phi$ if and only if there is $(t_1, \ldots, t_n) \in R$ such that $(S_1, \ldots, S_m) = \overline{[t_1, \ldots, t_n]}$.*

Proof. Let $\mathcal{A}$ be the automaton which accepts the set of terms $[t_1, \ldots, t_n]$ for $(t_1, \ldots, t_n) \in R$. The terminal alphabet of $\mathcal{A}$ is $\mathcal{F}' = (\mathcal{F} \cup \{\perp\})^n$, the set of states Q, the final states Q_f and the set of transition rules T. Let $\mathcal{F}' = \{f_1, \ldots, f_m\}$ and $Q = \{q_1, \ldots, q_l\}$. The following formula $\phi_{\mathcal{A}}$ (with $m+1$ free variables) defines the set $\{[t_1, \ldots, t_n] \mid (t_1, \ldots, t_n) \in R\}$.

$$\begin{array}{ll} \multicolumn{2}{l}{\exists Y_{q_1}, \ldots, \exists Y_{q_l}.} \\ & \mathrm{Term}(X, X_{f_1}, \ldots, X_{f_m}) \\ \wedge & \mathrm{Partition}(X, Y_{q_1}, \ldots, Y_{q_l}) \\ \wedge & \bigvee_{q \in Q_f} \epsilon \in Y_q \\ \wedge & \forall x. \bigwedge_{f \in \mathcal{F}'} \bigwedge_{q \in Q} ((x \in X_f \wedge x \in Y_q) \Rightarrow (\bigvee_{f(q_1,\ldots,q_s) \to q \in T} \bigwedge_{i=1}^{s} xi \in Y_{q_i})) \end{array}$$

This formula basically expresses that there is a successful run of the automaton on $[t_1, \ldots, t_n]$: the variables Y_{q_i} correspond to sets of positions which are labeled with q_i by the run. They should be a partition of the set of positions. The root has to be labeled with a final state (the run is successful). Finally, the last line expresses local properties that have to be satisfied by the run: if the sons xi of a position x are labeled with $q_1, ..., q_n$ respectively and x is labeled with symbol f and state q, then there should be a transition $f(q_1, \ldots, q_n) \to q$ in the set of transitions.

We have to prove two inclusions. First assume that $(S, S_1, \ldots, S_m) \models_2 \phi$. Then $(S, S_1, \ldots, S_m) \models \mathrm{Term}(X, X_{f_1}, \ldots, X_{f_m})$, hence there is a term $u \in T(\mathcal{F})'$ whose set of position is S and such that for all i, S_i is the set of positions labeled with f_i. Now, there is a partition $E_{q_1}, \ldots, E_{q_l}$ of S which satisfies

$$\begin{array}{l} S, S_1, \ldots, S_m, E_{q_1}, \ldots, E_{q_l} \models \\ \qquad \forall x. \bigwedge_{f \in \mathcal{F}'} \bigwedge_{q \in Q} ((x \in X_f \wedge x \in Y_q) \Rightarrow (\bigvee_{f(q_1,\ldots,q_s) \to q \in T} \bigwedge_{i=1}^{s} xi \in Y_{q_i})) \end{array}$$

This implies that the labeling $E_{q_1}, \ldots, E_{q_l}$ is compatible with the transition rules: it defines a run of the automaton. Finally, the condition that the root Λ belongs to E_{q_f} for some final state q_f implies that the run is successful, hence that u is accepted by the automaton.

Conversely, if u is accepted by the automaton, then there is a successful run of $\mathcal{A}$ on u and we can label its positions with states in such a way that this labeling is compatible with the transition rules in $\mathcal{A}$. □

Putting together Lemmas 2 and 3, we can state the following slogan (which is not very precise; the precise statements are given by the lemmas):

Theorem 21. *L is definable if and only if L is in Rec.*

And, as a consequence:

Theorem 22 ([TW68]). *WSkS is decidable.*

Proof. Given a formula ϕ of WSkS, by Lemma 2, we can compute a finite tree automaton which has the same solutions as ϕ. Now, assume that ϕ has no free variable. Then the alphabet of the automaton is empty (or, more precisely, it contains the only constant $\top$ according to what we explained in Section 3.2.4). Finally, the formula is valid iff the constant $\top$ is in the language, i.e. iff there is a rule $\top \rightarrow q_f$ for some $q_f \in Q_f$. $\square$

3.3.7 Complexity issues

We have seen in chapter 1 that, for finite tree automata, emptiness can be decided in linear time (and is PTIME-complete) and that inclusion is EXPTIME-complete. Considering WSkS formulas with a fixed number of quantifiers alternations N, the decision method sketched in the previous section will work in time which is a tower of exponentials, the height of the tower being $O(N)$. This is so because each time we encounter a sequence $\forall X, \exists Y$, the existential quantification corresponds to a projection, which may yield a non-deterministic automaton, even if the input automaton was deterministic. Then the elimination of $\forall X$ requires a determinization (because we have to compute a complement automaton) which requires in general exponential time and exponential space.

Actually, it is not really possible to do much better since, even when $k = 1$, deciding a formula of WSkS requires non-elementary time, as shown in [SM73].

3.3.8 Extensions

There are several extensions of the logic, which we already mentioned: though quantification is restricted to finite sets, we may consider infinite sets as models (this is what is often called *weak second-order monadic logic with k successors* and also written WSkS), or consider quantifications on arbitrary sets (this is the full SkS).

These logics require more sophisticated automata which recognize sets of *infinite* terms. The proof of Theorem 22 carries over these extensions, with the provision that the devices enjoy the required closure and decidability properties. But this becomes much more intricate in the case of infinite terms. Indeed, for infinite terms, it is not possible to design bottom-up tree automata. We have to use a top-down device. But then, as mentioned in chapter 1, we cannot expect to reduce the non-determinism. Now, the closure by complement becomes problematic because the usual way of computing the complement uses reduction of non-determinism as a first step.

It is out of the scope of this book to define and study automata on infinite objects (see [Tho90] instead). Let us simply mention that the closure under complement for *Rabin automata* which work on infinite trees (this result is known as *Rabin's Theorem*) is one of the most difficult results in the field

3.4 Examples of applications

3.4.1 Terms and sorts

The most basic example is what is known in the algebraic specification community as *order-sorted signatures* . These signatures are exactly what we called bottom-up tree automata. There are only differences in the syntax. For instance, the following signature:

SORTS:$\mathsf{Nat}, \mathsf{int}$
SUBSORTS : $\mathsf{Nat} \leq \mathsf{int}$
FUNCTION DECLARATIONS:

$$\begin{array}{lrcl}
0: & & \rightarrow & \mathsf{Nat} \\
+: & \mathsf{Nat}\times\mathsf{Nat} & \rightarrow & \mathsf{Nat} \\
s: & \mathsf{Nat} & \rightarrow & \mathsf{Nat} \\
p: & \mathsf{Nat} & \rightarrow & \mathsf{int} \\
+: & \mathsf{int}\times\mathsf{int} & \rightarrow & \mathsf{int} \\
abs: & \mathsf{int} & \rightarrow & \mathsf{Nat} \\
fact: & \mathsf{Nat} & \rightarrow & \mathsf{Nat} \\
\ldots & & &
\end{array}$$

is an automaton whose states are $\mathsf{Nat}, \mathsf{int}$ with an ϵ-transition from Nat to int and each function declaration corresponds to a transition of the automaton. For example $+(\mathsf{Nat}, \mathsf{Nat}) \rightarrow \mathsf{Nat}$. The set of *well-formed terms* (as in the algebraic specification terminology) is the set of terms recognized by the automaton in any state.

More general typing systems also correspond to more general automata (as will be seen e.g. in the next chapter).

This correspondence is not surprising; types and sorts are introduced in order to prevent run-time errors by some "abstract interpretation" of the inputs. Tree automata and tree grammars also provide such a symbolic evaluation mechanism. For other applications of tree automata in this direction, see e.g. chapter 5.

From what we have seen in this chapter, we can go beyond simply recognizing the set of well-formed terms. Consider the following *sort constraints* (the alphabet $\mathcal{F}$ of function symbols is given):

The set of *sort expressions* $\mathcal{SE}$ is the least set such that

- $\mathcal{SE}$ contains a finite set of *sort symbols* S, including the two particular symbols $\top_S$ and $\bot_S$.
- If $s_1, s_2 \in \mathcal{SE}$, then $s_1 \vee s_2$, $s_1 \wedge s_2$, $\neg s_1$ are in $\mathcal{SE}$
- If $s_1, \ldots, s_n$ are in $\mathcal{SE}$ and f is a function symbol of arity n, then $f(s_1, \ldots, s_n) \in \mathcal{SE}$.

The atomic formulas are expressions $t \in s$ where $t \in T(\mathcal{F}, \mathcal{X})$ and $s \in \mathcal{SE}$. The formulas are arbitrary first-order formulas built on these atomic formulas.

These formulas are interpreted as follows: we are giving an order-sorted signature (or a tree automaton) whose set of sorts is S. We define the interpretation$[\![\cdot]\!]_S$ of sort expressions as follows:

- if $s \in S$, $[\![s]\!]_S$ is the set of terms in $T(\mathcal{F})$ that are accepted in state s.
- $[\![\top_S]\!]_S = T(\mathcal{F})$ and $[\![\bot_S]\!]_S = \emptyset$
- $[\![s_1 \vee s_2]\!]_S = [\![s_1]\!]_S \cup [\![s_2]\!]_S$, $[\![s_1 \wedge s_2]\!]_S = [\![s_1]\!]_S \cap [\![s_2]\!]_S$, $[\![\neg s]\!]_S = T(\mathcal{F}) \setminus [\![s]\!]_S$
- $[\![f(s_1, \ldots, s_n)]\!]_S = \{f(t_1, \ldots, t_n) \mid t_1 \in [\![s_1]\!]_S, \ldots t_n \in [\![s_n]\!]_S\}$

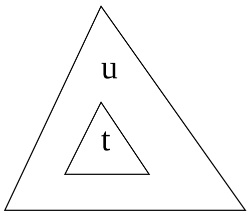


Figure 3.11: u encompasses t

An assignment σ, mapping variables to terms in $T(\mathcal{F})$, *satisfies* $t \in s$ (we also say that σ *is a solution* of $t \in s$) if $t\sigma \in [\![s]\!]_S$. Solutions of arbitrary formulas are defined as expected. Then

Theorem 23. *Sort constraints are decidable.*

The decision technique is based on automata computations, following the closure properties of $Rec_\times$ and a decomposition lemma for constraints of the form $f(t_1, \dots, t_n) \in s$.

More results and applications of sort constraints are discussed in the bibliographic notes.

3.4.2 The encompassment theory for linear terms

Definition 9. *If $t \in T(\mathcal{F}, \mathcal{X})$ and $u \in T(\mathcal{F})$, u* encompasses *t if there is a substitution σ such that $t\sigma$ is a subterm of u. (See Figure 3.11.) This binary relation is denoted $t \trianglelefteq u$ or, seen as a unary relation on ground terms parametrized by t: $\trianglelefteq_t(u)$.*

Encompassment plays an important role in rewriting: a term t is reducible by a term rewriting system R if and only if t encompasses at least one left hand side of a rule.

The relationship with tree automata is given by the proposition:

Proposition 15. *If t is linear, then the set of terms that encompass t is recognized by an NFTA of size $O(|t|)$.*

Proof. To each non-variable subterm v of t we associate a state q_v. In addition we have a state $q_\top$. The only final state is q_t. The transition rules are:

- $f(q_\top, \dots, q_\top) \to q_\top$ for all function symbols.
- $f(q_{t_1}, \dots, q_{t_n}) \to q_{f(t_1,\dots,t_n)}$ if $f(t_1, \dots, t_n)$ is a subterm of t and q_{t_i} is actually $q_\top$ is t_i is a variable.
- $f(q_\top \dots, q_\top, q_t, q_\top, \dots, q_\top) \to q_t$ for all function symbols f whose arity is at least 1.

The proof that this automaton indeed recognizes the set of terms that encompass t is left to the reader. □

Note that the automaton may be non deterministic.

Corollary 4. *If $\mathcal{R}$ is a term rewriting system whose all left members are linear, then the set of reducible terms in $T(\mathcal{F})$, as well as the set of irreducible terms in $T(\mathcal{F})$ are recognized by a finite tree automaton.*

Proof. This is a consequence of Theorem 5. □

The *theory of reducibility* associated with a set of term $S \subseteq T(\mathcal{F}, \mathcal{X})$ is the set of first-order formulas built on the unary predicate symbols E_t, $t \in S$ and interpreted as the set of terms encompassing t.

Theorem 24. *The reducibility theory associated with a set of linear terms is decidable.*

Proof. By Proposition 14, the set of solutions of an atomic formula is recognizable, hence definable in WSkS by Lemma 3. Hence, any first-order formula built on these atomic predicate symbols can be translated into a (second-order) formula of WSkS which has the same models (up to the coding of terms into tuples of sets). Then, by Theorem 22, the reducibility theory associated with a set of linear terms is decidable. □

Note however that we do not use here the full power of WSkS. Actually, the solutions of a Boolean combination of atomic formulas are in $Rec_{\times}$. So, we cannot apply the complexity results for WSkS here. (In fact, the complexity of the reducibility theory is unknown so far).

Let us simply show an example of an interesting property of rewrite systems which can be expressed in this theory.

Definition 10. *Given a term rewriting system R, a term t is* ground reducible *if, for every ground substitution σ, $t\sigma$ is reducible by R.*

Note that a term might be irreducible and still ground reducible. For instance consider the alphabet $\mathcal{F} = \{0, s\}$ and the rewrite system $R = \{s(s(0)) \to 0\}$. Then the term $s(s(x)$ is irreducible by R, but all its ground instances are reducible.

It turns out that ground reducibility of t is expressible in the encompassment theory by the formula:

$$\forall x.(\trianglelefteq_t(x) \Rightarrow \bigvee_{i=1}^{n} \trianglelefteq_{l_i}(x))$$

Where $l_1, \ldots, l_n$ are the left hand sides of the rewrite system. By Theorem 24, if $t, l_1, \ldots, l_n$ are linear, then ground reducibility is decidable. Actually, it has been shown that this problem is EXPTIME-complete, but is beyond the scope of this book to give the proof.

3.4.3 The first-order theory of a reduction relation: the case where no variables are shared

We consider again an application of tree automata to decision problem in logic and term rewriting.

Consider the following logical theory. Let $\mathcal{L}$ be the set of all first-order formulas using no function symbols and a single binary predicate symbol $\rightarrow$.

Given a rewrite system $\mathcal{R}$, interpreting $\rightarrow$ as $\xrightarrow[\mathcal{R}]{}$, yields the *theory of one step rewriting*; interpreting $\rightarrow$ as $\xrightarrow[\mathcal{R}]{*}$ yields the *theory of rewriting.*

Both theories are undecidable for arbitrary $\mathcal{R}$. They become however decidable if we restrict our attention to term rewriting systems in which each variable occurs at most once. Basically, the reason is given by the following:

Proposition 16. *If $\mathcal{R}$ is a linear rewrite system such that left and right members of the rules do not share variables, then $\xrightarrow[\mathcal{R}]{*}$ is recognized by a GTT.*

Proof. As in the proof of Proposition 15, we can construct in linear time a (non-deterministic) automaton which accepts the set of instances of a linear term. For each rule $l_i \rightarrow r_i$ we can construct a pair $(\mathcal{A}_i, \mathcal{A}'_i)$ of automata which respectively recognize the set of instances of l_i and the set of instances of r_i. Assume that the sets of states of the $\mathcal{A}_i$s are pairwise disjoint and that each $\mathcal{A}_i$ has a single final state q^i_f. We may assume a similar property for the $\mathcal{A}'_i$s: they do not share states and for each i, the only common state between $\mathcal{A}_i$ and $\mathcal{A}'_i$ is q^i_f (the final state for both of them). Then $\mathcal{A}$ (resp. $\mathcal{A}'$) is the union of the $\mathcal{A}_i$s: the states are the union of all sets of states of the $\mathcal{A}_i$s (resp. $\mathcal{A}'_i$s), transitions and final states are also unions of the transitions and final states of each individual automaton.

We claim that $(\mathcal{A}, \mathcal{A}')$ defines a GTT whose closure by iteration $(\mathcal{A}_*, \mathcal{A}'_*)$ (which is again a GTT according to Theorem 20) accepting $\xrightarrow[\mathcal{R}]{*}$. For, assume first that $u \xrightarrow[l_i \rightarrow r_i]{p} v$. Then $u|_p$ is an instance $l_i\sigma$ of l_i, hence is accepted in state q^i_f. $v|_p$ is an instance $r_i\theta$ of r_i, hence accepted in state q^i_f. Now, $v = u[r_i\theta]_p$, hence (u, v) is accepted by the GTT $(\mathcal{A}, \mathcal{A}')$. It follows that if $u \xrightarrow[\mathcal{R}]{*} v$, (u, v) is accepted by $(\mathcal{A}_*, \mathcal{A}'_*)$.

Conversely, assume that (u, v) is accepted by $(\mathcal{A}, \mathcal{A}')$, then

$$u \xrightarrow[\mathcal{A}]{*} C[q_1, \dots, q_n]_{p_1, \dots, p_n} \xleftarrow[\mathcal{A}']{*} v$$

Moreover, each q_i is some state q^j_f, which, by definition, implies that $u|_{p_i}$ is an instance of l_j and $v|_{p_i}$ is an instance of r_j. Now, *since l_j and r_j do not share variables*, for each i, $u|_{p_i} \xrightarrow[\mathcal{R}]{} v|_{p_i}$. Which implies that $u \xrightarrow[\mathcal{R}]{*} v$. Now, if (u, v) is accepted by $(\mathcal{A}_*, \mathcal{A}'_*)$, u can be rewritten in v by the transitive closure of $\xrightarrow[\mathcal{R}]{*}$, which is $\xrightarrow[\mathcal{R}]{*}$ itself.

□

Theorem 25. *If $\mathcal{R}$ is a linear term rewriting system such that left and right members of the rules do not share variables, then the first-order theory of rewriting is decidable.*

Proof. By Proposition 16, $\xrightarrow[\mathcal{R}]{*}$ is recognized by a GTT. From Proposition 9, $\xrightarrow[\mathcal{R}]{*}$ is in *Rec*. By Lemma 3, there is a WSkS formula whose solutions are exactly the pairs (s, t) such that $s \xrightarrow[\mathcal{R}]{*} t$. Finally, by Theorem 22, the first-order theory of $\xrightarrow[\mathcal{R}]{*}$ is decidable. □

3.4.4 Reduction strategies

So far, we gave examples of first-order theories (or *constraint systems*) which can be decided using tree automata techniques. Other examples will be given in the next two chapters. We give here another example of application in a different spirit: we are going to show how to decide the existence (and compute) "optimal reduction strategies" in term rewriting systems. Informally, a reduction sequence is optimal when every redex which is contracted along this sequence has to be contracted in any reduction sequence yielding a normal form. For example, if we consider the rewrite system $\{x \vee \top \to \top; \top \vee x \to \top\}$, there is no optimal sequential reduction strategy in the above sense since, given an expression $e_1 \vee e_2$, where e_1 and e_2 are unevaluated, the strategy should specify which of e_1 or e_2 has to be evaluated first. However, if we start with e_1, then maybe e_2 will reduce to $\top$ and the evaluation step on e_1 was unnecessary. Symmetrically, evaluating e_2 first may lead to unnecessary computations. An interesting question is to give sufficient criteria for a rewrite system to admit optimal strategies and, in case there is such a strategy, give it explicitly.

The formalization of these notions was given by Huet and Lévy in [HL91] who introduce the notion of *sequentiality*. We give briefly a summary of (part of) their definitions.

$\mathcal{F}$ is a fixed alphabet of function symbols and $\mathcal{F}_\Omega = \mathcal{F} \cup \{\Omega\}$ is the alphabet $\mathcal{F}$ enriched with a new constant Ω (whose intended meaning is "unevaluated term").

We define on $T(\mathcal{F}_\Omega)$ the relation "less evaluated than" as:

> $u \sqsubseteq v$ if and only if either $u = \Omega$ or else $u = f(u_1, \ldots, u_n)$, $v = f(v_1, \ldots, v_n)$ and for all i, $u_i \sqsubseteq v_i$

Definition 11. *A predicate P on $T(\mathcal{F}_\Omega)$ is* monotonic *if $u \in P$ and $u \sqsubseteq v$ implies $v \in P$.*

For example, a monotonic predicate of interest for rewriting is the predicate $N_\mathcal{R}$: $t \in N_\mathcal{R}$ if and only if there is a term $u \in T(\mathcal{F})$ such that u is irreducible by $\mathcal{R}$ and $t \xrightarrow[\mathcal{R}]{*} u$.

Definition 12. *Let P be a monotonic predicate on $T(\mathcal{F}_\Omega)$. If $\mathcal{R}$ is a term rewriting system and $t \in T(\mathcal{F}_\Omega)$, a position p of Ω in t is an* index *for P if for all terms $u \in T(\mathcal{F}_\Omega)$ such that $t \sqsubseteq u$ and $u \in P$, then $u|_p \neq \Omega$*

In other words: it is necessary to evaluate t at position p in order to have the predicate P satisfied.

Example 32. Let $\mathcal{R} = \{f(g(x), y) \to g(f(x, y));\ f(a, x) \to a; b \to g(b)\}$. Then 1 is an index of $f(\Omega, \Omega)$ for $N_\mathcal{R}$: any reduction yielding a normal form without Ω will have to evaluate the term at position 1. More formally, every term $f(t_1, t_2)$ which can be reduced to a term in $T(\mathcal{F})$ in normal form satisfies $t_1 \neq \Omega$. On the contrary, 2 is not an index of $f(\Omega, \Omega)$ since $f(a, \Omega) \xrightarrow[R]{*} a$.

Definition 13. *A monotonic predicate P is* sequential *if every term t such that:*

- $t \notin P$
- *there is $u \in T(\mathcal{F})$, $t \sqsubseteq u$ and $u \in P$*

has an index for P.

If $N_{\mathcal{R}}$ is sequential, the reduction strategy consisting of reducing an index is optimal for non-overlapping and left linear rewrite systems.

Now, the relationship with tree automata is given by the following result:

Theorem 26. *If P is definable in WSkS, then the sequentiality of P is also definable in WSkS.*

The proof of this result is quite easy: it suffices to translate directly the definitions.

For instance, if $\mathcal{R}$ is a rewrite system whose left and right members do not share variables, then $N_{\mathcal{R}}$ is recognizable (by Propositions 16 and 9), hence definable in WSkS by Lemma 3 and the sequentiality of $\mathcal{R}$ is decidable by Theorem 26.

In general, the sequentiality of $N_{\mathcal{R}}$ is undecidable. However, one can notice that, if $\mathcal{R}$ and $\mathcal{R}'$ are two rewrite systems such that $\xrightarrow[\mathcal{R}]{} \subseteq \xrightarrow[\mathcal{R}']{}$, then a position p which is an index for $\mathcal{R}'$ is also an index for $\mathcal{R}$. (And thus, $\mathcal{R}$ is sequential whenever $\mathcal{R}'$ is sequential).

For instance, we may approximate the term rewriting system, replacing all right hand sides by a new variable which does not occur in the corresponding left member. Let $\mathcal{R}?$ be this approximation and $N?$ be the predicate $N_{\mathcal{R}?}$. (This is the approximation considered by Huet and Lévy).

Another, refined, approximation consists in renaming all variables of the right hand sides of the rules in such a way that all right hand sides become linear and do not share variables with their left hand sides. Let $\mathcal{R}'$ be such an approximation of $\mathcal{R}$. The predicate $NF_{\mathcal{R}'}$ is written NV.

Proposition 17. *If $\mathcal{R}$ is left linear, then the predicates $NF?$ and NV are definable in WSkS and their sequentiality is decidable.*

Proof. The approximations $\mathcal{R}?$ and $\mathcal{R}'$ satisfy the hypotheses of Proposition 16 and hence $\xrightarrow[\mathcal{R}?]{*}$ and $\xrightarrow[\mathcal{R}']{*}$ are recognized by GTTs. On the other hand, the set of terms in normal form for a left linear rewrite system is recognized by a finite tree automaton (see Corollary 4). By Proposition 9 and Lemma 3, all these predicates are definable in WSkS. Then $N?$ and NV are also definable in WSkS. For instance for NV :

$$NV(t) \stackrel{\text{def}}{=} \exists u.t \xrightarrow[\mathcal{R}']{*} u \wedge NF(u)$$

Then, by Theorem 26, the sequentiality of $N?$ and NV are definable in WSkS and by Theorem 22 they are decidable. $\square$

3.4.5 Application to rigid E-unification

Given a (universally quantified) set of equations E, the classical problem of E-*unification* is, given an equation $s = t$, find substitutions σ such that $E \models s\sigma = t\sigma$. The associated decision problem is to decide whether such a substitution exists. This problem is in general unsolvable: there are decision procedures for restricted classes of axioms E.

The *simultaneous rigid E-unification problem* is slightly different: we are still giving E and a finite set of equations $s_i = t_i$ and the question is to find a substitution σ such that

$$\models (\bigwedge_{e \in E} e\sigma) \Rightarrow (\bigwedge_{i=1}^{n} s_i\sigma = t_i\sigma)$$

The associated decision problem is to decide the existence of such a substitution.

The relevance of such questions to automated deduction is very briefly described in the bibliographic notes. We want here to show how tree automata help in this decision problem.

Simultaneous rigid E-unification is undecidable in general. However, for the one variable case, we have:

Theorem 27. *The simultaneous rigid E-unification problem with one variable is EXPTIME-complete.*

The EXPTIME membership is a direct consequence of the following lemma, together with closure and decision properties for recognizable tree languages. The EXPTIME-hardness is obtained by reduction the intersection non-emptiness problem, see Theorem 11).

Lemma 4. *The solutions of a rigid E-unification problem with one variable are recognizable by a finite tree automaton.*

Proof. (sketch) Assume that we have a single equation $s = t$. Let x be the only variable occurring in $E, s = t$ and $\hat{E}$ be the set E in which x is considered as a constant. Let R be a canonical ground rewrite system associated with $\hat{E}$ (and for which x is minimal). We define v as x if s and t have the same normal form w.r.t. R and as the normal form of $x\sigma$ w.r.t. R otherwise.

Assume $E\sigma \models s\sigma = t\sigma$. If $v \not\equiv x$, we have $\hat{E} \cup \{x = v\} \models x = x\sigma$. Hence $\hat{E} \cup \{x = v\} \models s = t$ in any case. Conversely, assume that $\hat{E} \cup \{x = v\} \models s = t$. Then $\hat{E} \cup \{x = x\sigma\} \models s = t$, hence $E\sigma \models s\sigma = t\sigma$.

Now, assume $v \not\equiv x$. Then either there is a subterm u of an equation in $\hat{E}$ such that $\hat{E} \models u = v$ or else $R_1 = R \cup \{v \to x\}$ is canonical. In this case, from $\hat{E} \cup \{v = x\} \models s = t$, we deduce that either $\hat{E} \models s = t$ (and $v \equiv x$) or there is a subterm u of s, t such that $\hat{E} \models v = u$. we can conclude that, in all cases, there is a subterm u of $E \cup \{s = t\}$ such that $\hat{E} \models u = v$.

To summarize, σ is such that $E\sigma \models s\sigma = t\sigma$ iff there is a subterm u of $E \cup \{s = t\}$ such that $\hat{E} \models u = x\sigma$ and $\hat{E} \cup \{u = x\} \models s = t$.

If we let T be the set of subterms of $E \cup \{s = t\}$ such that $\hat{E} \cup \{u = x\} \models s = t$, then T is finite (and computable in polynomial time). The set of solutions is then $\xrightarrow[R^{-1}]{*}(T)$, which is a recognizable set of trees, thanks to Proposition 16. □

Further comments and references are given in the bibliographic notes.

3.4.6 Application to higher-order matching

We give here a last application (but the list is not closed!), in the typed lambda-calculus.

To be self-contained, let us first recall some basic definitions in typed lambda calculus.

The set of *types* of the simply typed lambda calculus is the least set containing the constant o (basic type) and such that $\tau \rightarrow \tau'$ is a type whenever τ and τ' are types.

Using the so-called Curryfication, any type $\tau \rightarrow (\tau' \rightarrow \tau'')$ is written $\tau, \tau' \rightarrow \tau''$. In this way all non-basic types are of the form $\tau_1, \ldots, \tau_n \rightarrow o$ with intuitive meaning that this is the type of functions taking n arguments of respective types $\tau_1, \ldots, \tau_n$ and whose result is a basic type o.

The ***order*** of a type τ is defined by:

- $O(o) = 1$
- $O(\tau_1, \ldots, \tau_n \rightarrow o) = 1 + \max\{O(\tau_1), \ldots, O(\tau_n)\}$

Given, for each type τ a set of variables $\mathcal{X}_\tau$ of type τ and a set C_τ of constants of type τ, the set of ***terms*** (of the simply typed lambda calculus) is the least set Λ such that:

- $x \in \mathcal{X}_\tau$ is a term of type τ
- $c \in C_\tau$ is a term of type τ
- If $x_1 \in \mathcal{X}_{\tau_1}, \ldots, x_n \in \mathcal{X}_{\tau_n}$ and t is a term of type τ, then $\lambda x_1, \ldots x_n : t$ is a term of type $\tau_1, \ldots, \tau_n \rightarrow \tau$
- If t is a term of type $\tau_1, \ldots, \tau_n \rightarrow \tau$ and $t_1, \ldots, t_n$ are terms of respective types $\tau_1, \ldots, \tau_n$, then $t(\tau_1, \ldots, \tau_n)$ is a term of type τ.

The ***order*** of a term t is the order of its type $\tau(t)$.

The set of ***free variables*** $\mathcal{V}ar(t)$ of a term t is defined by:

- $\mathcal{V}ar(x) = \{x\}$ if x is a variable
- $\mathcal{V}ar(c) = \emptyset$ if c is a constant
- $\mathcal{V}ar(\lambda x_1 \ldots, x_n : t) = \mathcal{V}ar(t) \setminus \{x_1, \ldots, x_n\}$
- $\mathcal{V}ar(t(u_1, \ldots, u_n)) = \mathcal{V}ar(t) \cup \mathcal{V}ar(t_1) \cup \ldots \cup \mathcal{V}ar(t_n)$

Terms are always assumed to be in ***η-long form***, i.e. they are assumed to be in normal form with respect to the rule:

$$(\eta) \quad t \rightarrow \lambda x_1 \ldots x_n . t(x_1, \ldots, x_n) \quad \begin{array}{l} \text{if } \tau(t) = \tau_1, \ldots \tau_n \rightarrow \tau \\ \text{and } x_i \in \mathcal{X}_{\tau_i} \setminus \mathcal{V}ar(t) \text{ for all } i \end{array}$$

We define the ***α-equivalence*** $=_\alpha$ on Λ as the least congruence relation such that: $\lambda x_1, \ldots, x_n : t =_\alpha \lambda x'_1 \ldots, x'_n : t'$ when

- t' is the term obtained from t by substituting for every index i, every free occurrence of x_i with x'_i.

- There is no subterm of t in which, for some index i, both x_i and x'_i occur free.

In the following, we consider only lambda terms modulo α-equivalence. Then we may assume that, in any term, any variable is bounded at most once and free variables do not have bounded occurrences.

The ***β-reduction*** is defined on Λ as the least binary relation $\underset{\beta}{\rightarrow}$ such that

- $\lambda x_1 \dots x_n : t(t_1, \dots, t_n) \underset{\beta}{\rightarrow} t\{x_1 \leftarrow t_1, \dots, x_n \leftarrow t_n\}$
- for every context C, $C[t] \underset{\beta}{\rightarrow} C[u]$ whenever $t \underset{\beta}{\rightarrow} u$

It is well-known that $\beta\eta$-reduction is terminating and confluent on Λ and, for every term $t \in \Lambda$, we let $t\downarrow$ be the unique normal form of t.

A ***matching problem*** is an equation $s = t$ where $s, t \in \Lambda$ and $\mathcal{V}ar(t) = \emptyset$. A ***solution*** of a matching problem is a substitution σ of the free variables of t such that $s\sigma \downarrow = t \downarrow$.

Whether or not the matching problem is decidable is an open question at the time we write this book. However, it can be decided when every free variable occurring in s is of order less or equal to 4. We sketch here how tree automata may help in this matter. We will consider only two special cases here, leaving the general case as well as details of the proofs as exercises (see also bibliographic notes).

First consider a problem

$$(1) \qquad\qquad x(s_1, \dots, s_n) = t$$

where x is a third order variable and $s_1, \dots, s_n, t$ are terms without free variables.

The first result states that the set of solutions is recognizable by a $\Box$-automaton. $\Box$-automata are a slight extension of finite tree automata: we assume here that the alphabet contains a special symbol $\Box$. Then a term u is accepted by a $\Box$-automaton $\mathcal{A}$ if and only if there is a term v which is accepted (in the usual sense) by $\mathcal{A}$ and such that u is obtained from v by replacing each occurrence of the symbol $\Box$ with a term (of appropriate type). Note that two distinct occurrences of $\Box$ need not to be replaced with the same term.

We consider the automaton $\mathcal{A}_{s_1,\dots,s_n,t}$ defined by: $\mathcal{F}$ consists of all symbols occurring in t plus the variable symbols $x_1, \dots, x_n$ whose types are respectively the types of $s_1, \dots, s_n$ and the constant $\Box$.

The set of states Q consists of all subterms of t, which we write q_u (instead of u) and a state $q_\Box$. In addition, we have the final state q_f.

The transition rules Δ consist in

- The rules
$$f(q_{t_1}, \dots, q_{t_n}) \rightarrow q_{f(t_1,\dots,t_n)}$$
each time $q_{f(t_1,\dots,t_n)} \in Q$
- For $i = 1, \dots, n$, the rules
$$x_i(q_{t_1}, \dots, q_{t_n}) \rightarrow q_u$$
where u is a subterm of t such that $s_i(t_1, \dots, t_n) \downarrow = u$ and $t_j = \Box$ whenever $s_i(t_1, \dots, t_{j-1}, \Box, t_{j+1}, \dots, t_n) \downarrow = u$.

- the rule $\lambda x_1, \ldots, \lambda x_n . q_t \rightarrow q_f$

Theorem 28. *The set of solutions of (1) (up to α-conversion) is accepted by the $\Box$-automaton $\mathcal{A}_{s_1,\ldots,s_n,t}$.*

More about this result, its proof and its extension to fourth order will be given in the exercises (see also bibliographic notes). Let us simply give an example.

Example 33. Let us consider the interpolation equation

$$x(\lambda y_1 \lambda y_2 . y_1, \lambda y_3 . f(y_3, y_3)) = f(a, a)$$

where y_1, y_2 are assumed to be of base type o. Then $\mathcal{F} = \{a, f, x_1, \Box_o\}$. $Q = \{q_a, q_{f(a,a)}, q_{\Box_o}\}$ and the rules of the automaton are:

$$\begin{array}{rclrcl}
a & \rightarrow & q_a & f(q_a, q_a) & \rightarrow & q_{f(a,a)} \\
\Box_o & \rightarrow & q_{\Box_o} & x_1(q_a, q_{\Box_o}) & \rightarrow & q_a \\
x_1(q_{f(a,a)}, q_{\Box_o}) & \rightarrow & q_{f(a,a)} & x_2(q_a) & \rightarrow & q_{f(a,a)} \\
\lambda x_1 \lambda x_2 . q_{f(a,a)} & \rightarrow & q_f & & &
\end{array}$$

For instance the term $\lambda x_1 \lambda x_2 . x_1(x_2(x_1(x_1(a, \Box_o), \Box_o)), \Box_o)$ is accepted by the automaton :

$$\begin{array}{rcl}
\lambda x_1 \lambda x_2 . x_1(x_2(x_1(x_1(a, \Box_o), \Box_o)), \Box_o) & \xrightarrow[\mathcal{A}]{*} & \lambda x_1 \lambda x_2 . x_1(x_2(x_1(x_1(q_a, q_{\Box_o}), q_{\Box_o})), q_{\Box_o}) \\
& \xrightarrow[\mathcal{A}]{} & \lambda x_1 \lambda x_2 . x_1(x_2(x_1(q_a, q_{\Box_o})), q_{\Box_o}) \\
& \xrightarrow[\mathcal{A}]{} & \lambda x_1 \lambda x_2 . x_1(x_2(q_a), q_{\Box_o}) \\
& \xrightarrow[\mathcal{A}]{} & \lambda x_1 \lambda x_2 . x_1(q_{f(a,a)}, q_{\Box_o}) \\
& \xrightarrow[\mathcal{A}]{} & \lambda x_1 \lambda x_2 . q_{f(a,a)} \\
& \xrightarrow[\mathcal{A}]{} & q_f
\end{array}$$

And indeed, for every terms t_1, t_2, t_3, $\lambda x_1 \lambda x_2 . x_1(x_2(x_1(x_1(a, t_1), t_2)), t_3)$ is a solution of the interpolation problem.

3.5 Exercises

Exercise 31. Let $\mathcal{F}$ be the alphabet consisting of finitely many unary function symbols $a_1, \ldots, a_n$ and a constant 0.

1. Show that the set S of pairs (of words) $\{(a_1^n(a_1(a_2(a_2^p(0)))), a_1^n(a_2^p(0))) \mid n, p \in \mathbb{N}\}$ is in *Rec*. Show that S^* is not in *Rec*, hence that *Rec* is not closed under transitive closure.
2. More generally, show that, for any finite rewrite system $\mathcal{R}$ (on the alphabet $\mathcal{F}$!), the reduction relation $\xrightarrow[\mathcal{R}]{}$ is in *Rec*.
3. Is there any hope to design a class of tree languages which contains *Rec*, which is closed by all Boolean operations and by transitive closure and for which emptiness is decidable ? Why ?

Exercise 32. Show that the set of pairs $\{(t, f(t,t')) \mid t, t' \in T(\mathcal{F})\}$ is not in *Rec*.

Exercise 33. Show that if a binary relation is recognized by a GTT, then its inverse is also recognized by a GTT.

Exercise 34. Give an example of two relations that are recognized by GTTs and whose union is not recognized by any GTT.

Similarly, show that the class of relations recognized by a GTT is not closed by complement. Is the class closed by intersection ?

Exercise 35. Give an example of a n-ary relation such that its ith projection followed by its ith cylindrification does not give back the original relation. On the contrary, show that ith cylindrification followed by ith projection gives back the original relation.

Exercise 36. About *Rec* and bounded delay relations. We assume that $\mathcal{F}$ only contains unary function symbols and a constant, i.e. we consider words rather than trees and we write $u = a_1 \dots a_n$ instead of $u = a_1(\dots(a_n(0))\dots)$. Similarly, $u \cdot v$ corresponds to the usual concatenation of words.

A binary relation R on $T(\mathcal{F})$ is called a *bounded delay relation* if and only if

$$\exists k / \forall (u,v) \in R, \mid |u| - |v| \mid \leq k$$

R preserves the length if and only if

$$\forall (u,v) \in R, \ \ |u| = |v|$$

If A, B are two binary relations, we write $A \cdot B$ (or simply AB) the relation

$$A \cdot B \overset{\text{def}}{=} \{(u,v) / \exists (u_1,v_1) \in A, (u_2,v_2) \in B u = u_1.u_2, v = v_1.v_2\}$$

Similarly, we write (in this exercise !)

$$A^* = \{(u,v) / \exists (u_1,v_1) \in A, \dots, (u_n,v_n) \in A, u = u_1 \dots u_n, v = v_1 \dots v_n\}$$

1. Given $A, B \in$ *Rec*, is $A \cdot B$ necessary in *Rec* ? is A^* necessary in *Rec* ? Why ?
2. Show that if $A \in$ *Rec* preserves the length, then $A^* \in$ *Rec*.
3. Show that if $A, B \in$ *Rec* and A is of bounded delay, then $A \cdot B \in$ *Rec*.
4. The family of *rational relations* is the smallest set of subsets of $T(\mathcal{F})^2$ which contains the finite subsets of $T(\mathcal{F})^2$ and which is closed under union, concatenation $(\cdot)$ and $*$.

   Show that, if A is a bounded delay rational relation, then $A \in$ *Rec*. Is the converse true ?

Exercise 37. Let R_0 be the rewrite system $\{x \times 0 \to 0; 0 \times x \to 0\}$ and $\mathcal{F} = \{0, 1, s, \times\}$

1. Construct explicitly the GTT accepting $\xrightarrow[R_0]{*}$.
2. Let $R_1 = R_0 \cup \{x \times 1 \to x\}$. Show that $\xrightarrow[R_1]{*}$ is is not recognized by a GTT.
3. Let $R_2 = R_1 \cup \{1 \times x \to x \times 1\}$. Using a construction similar to the transitive closure of GTTs, show that the set $\{t \in T(\mathcal{F}) \mid \exists u \in T(\mathcal{F}), t \xrightarrow[R_2]{*} u, u \in NF\}$ where NF is the set of terms in normal form for R_2 is recognized by a finite tree automaton.

Exercise 38. (*) More generally, prove that given any rewrite system $\{l_i \rightarrow r_i \mid 1 \leq i \leq n\}$ such that

1. for all i, l_i and r_i are linear
2. for all i, if $x \in \mathcal{V}ar(l_i) \cap \mathcal{V}ar(r_i)$, then x occurs at depth at most one in l_i.

the set $\{t \in T(\mathcal{F}) \mid \exists u \in NF, t \xrightarrow[R]{*} u\}$ is recognized by finite tree automaton.

What are the consequences of this result ?

(See [Jac96] for details about this results and its applications. Also compare with Exercise 15, question 4.)

Exercise 39. Show that the set of pairs $\{(f(t,t'),t) \mid t,t' \in T(\mathcal{F})\}$ is not definable in WSkS. (See also Exercise 32)

Exercise 40. Show that the set of pairs of words $\{(w,w') \mid l(w) = l(w')\}$, where $l(x)$ is the length of x, is not definable in WSkS.

Exercise 41. Let $\mathcal{F} = \{a_1, \ldots, a_n, 0\}$ where each a_i is unary and 0 is a constant. Consider the following constraint system: terms are built on $\mathcal{F}$, the binary symbols $\cap, \cup$, the unary symbol $\neg$ and set variables. Formulas are conjunctions of inclusion constraints $t \subseteq t'$. The formulas are interpreted by assigning to variables finite subsets of $T(\mathcal{F})$, with the expected meaning for other symbols.

Show that the set of solutions of such constraints is in Rec_2. What can we conclude ?

(*) What happens if we remove the condition on the a_i's to be unary?

Exercise 42. Complete the proof of Proposition 13.

Exercise 43. Show that the subterm relation is not definable in WSkS.

Given a term t Write a WSkS formula ϕ_t such that a term $u \models \phi_t$ if and only if t is a subterm of u.

Exercise 44. Define in SkS "X is finite". (Hint: express that every totally ordered subset of X has an upper bound and use König's lemma)

Exercise 45. A tuple $(t_1, \ldots, t_n) \in T(\mathcal{F})^n$ can be represented in several ways as a finite sequence of finite sets. The first one is the encoding given in Section 3.3.6, overlapping the terms and considering one set for each tuple of symbols. A second one consists in having a tuple of sets for each component: one for each function symbol.

Compare the number of free variables which result from both codings when defining an n-ary relation on terms in WSkS. Compare also the definitions of the diagonal Δ using both encodings. How is it possible to translate an encoding into the other one ?

Exercise 46. (*) Let $\mathcal{R}$ be a finite rewrite system whose all left and right members are ground.

1. Let Termination(x) be the predicate on $T(\mathcal{F})$ which holds true on t when there is no infinite sequence of reductions starting from t. Show that adding this predicate as an atomic formula in the first-order theory of rewriting, this theory remains decidable for ground rewrite systems.
2. Generalize these results to the case where the left members of $\mathcal{R}$ are linear and the right members are ground.

Exercise 47. The complexity of automata recognizing the set of irreducible ground terms.

1. For each $n \in \mathbb{N}$, give a linear rewrite system $\mathcal{R}_n$ whose size is $O(n)$ and such that the minimal automaton accepting the set of irreducible ground terms has a size $O(2^n)$.

2. Assume that for any two strict subterms s, t of left hand side(s) of $\mathcal{R}$, if s and t are unifiable, then s is an instance of t or t is an instance of s. Show that there is a NFTA $\mathcal{A}$ whose size is linear in $\mathcal{R}$ and which accepts the set of irreducible ground terms.

Exercise 48. Prove Theorem 26.

Exercise 49. The Propositional Linear-time Temporal Logic. The logic **PTL** is defined as follows:

Syntax P is a finite set of *propositional variables.* Each symbol of P is a formula (an atomic formula). If ϕ and ψ are formulas, then the following are formulas:

$$\phi \wedge \psi,\ \phi \vee \psi,\ \phi \rightarrow \psi,\ \neg\phi,\ \phi\mathbf{U}\psi,\ \mathbf{N}\phi,\ \mathbf{L}\phi$$

Semantics Let P^* be the set of words over the alphabet P. A word $w \in P^*$ is identified with the sequence of letters $w(0)w(1)\dots w(|w|-1)$. $w(i..j)$ is the word $w(i)\dots w(j)$. The satisfaction relation is defined by:

- if $p \in P$, $w \models p$ if and only if $w(0) = p$
- The interpretation of logical connectives is the usual one
- $w \models \mathbf{N}\phi$ if and only if $|w| \geq 2$ and $w(1..|w|-1) \models \phi$
- $w \models \mathbf{L}\phi$ if and only if $|w| = 1$
- $w \models \phi\mathbf{U}\psi$ if and only if there is an index $i \in [0..|w|]$ such that for all $j \in [0..i]$, $w(j..|w|-1) \models \phi$ and $w(i..|w|-1) \models \psi$.

Let us recall that the language defined by a formula ϕ is the set of words w such that $w \models \phi$.

1. What it is the language defined by $\mathbf{N}(p_1\mathbf{U}p_2)$ (with $p_1, p_2 \in P$) ?
2. Give **PTL** formulas defining respectively $P^*p_1P^*$, p_1^*, $(p_1p_2)^*$.
3. Give a first-order WS1S formula (i.e. without second-order quantification and containing only one free second-order variable) which defines the same language as $\mathbf{N}(p_1\mathbf{U}p_2)$
4. For any **PTL** formula, give a first-order WS1S formula which defines the same language.
5. Conversely, show that any language defined by a first-order WS1S formula is definable by a **PTL** formula.

Exercise 50. About 3rd-order interpolation problems

1. Prove Theorem 28.
2. Show that the size of the automaton $\mathcal{A}_{s_1,\dots,s_n,t}$ is $O(n \times |t|)$
3. Deduce from Exercise 18 that the existence of a solution to a system of interpolation equations of the form $x(s_1, \dots, s_n) = t$ (where x is a third order variable in each equation) is in NP.

Exercise 51. About general third order matching.

1. How is it possible to modify the construction of $\mathcal{A}_{s_1,\dots,s_n,t}$ so as to forbid some symbols of t to occur in the solutions ?

2. Consider a third order matching problem $u = t$ where t is in normal form and does not contain any free variable. Let $x_1, \dots, x_n$ be the free variables of u and $x_i(s_1, \dots, s_m)$ be the subterm of u at position p. Show that, for every solution σ, either $u[\Box]_p\sigma \downarrow =_\alpha t$ or else that $x_i\sigma(s_1\sigma, \dots, s_m\sigma) \downarrow$ is in the set S_p defined as follows: $v \in S_p$ if and only if there is a subterm t' of t and there are positions $p_1, \dots, p_k$ of t' and variables $z_1, \dots, z_k$ which are bound above p in u such that $v = t'[z_1, \dots, z_k]_{p_1, \dots, p_k}$.
3. By guessing the results of $x_i\sigma(s_1\sigma, \dots, s_m\sigma)$ and using the previous exercise, show that general third order matching is in NP.

Exercise 52. About fourth-order matching.
** a completer **

3.6 Bibliographic Notes

The following bibliographic notes only concern the applications of the usual finite tree automata on finite trees (as defined at this stage of the book). We are pretty sure that there are many missing references and we are pleased to receive more pointers to the litterature.

3.6.1 GTT

GTT were introduced in [DTHL87] where they were used for the decidability of confluence of ground rewrite systems.

3.6.2 Automata and Logic

The development of automata in relation with logic and verification (in the sixties) is reported in [Tra95]. This research program was explained by A. Church himself in 1962 [Chu62].

Milestones of the decidability of monadic second-order logic are the papers [Büc60] [Rab69]. Theorem 22 is proved in [TW68].

3.6.3 Surveys

There are numerous surveys on automata and logic. Let us mention some of them: M.O. Rabin [Rab77] surveys the decidable theories; W. Thomas [Tho90, Tho97] provides an excellent survey of relationships between automata and logic.

3.6.4 Applications of tree automata to constraint solving

Concerning applications of tree automata, the reader is also referred to [Dau94] which reviews a number of applications of Tree automata to rewriting and constraints.

The relation between sorts and tree automata is pointed out in [Com89]. The decidability of arbitrary first-order sort constraints (and actually the first order theory of finite trees with equality and sort constraints) is proved in [CD94].

More general sort constraints involving some second-order terms are studied in [Com98b] with applications to a sort constrained equational logic [Com98a].

Sort constraints are also applied to specifications and automated inductive proofs in [BJ97] where tree automata are used to represent some normal forms sets. They are used in logic programming and automated reasoning [FSVY91, GMW97], in order to get more efficient procedures for fragments which fall into the scope of tree automata techniques. They are also used in automated deduction in order to increase the expressivity of (counter-)model constructions [Pel97].

Concerning encompassment, M. Dauchet et al gave a more general result (dropping the linearity requirement) in [DCC95]. We will come back to this result in the next chapter.

3.6.5 Application of tree automata to semantic unification

Rigid unification was originally considered by J. Gallier et al. [GRS87] who showed that this is a key notion in extending the matings method to a logic with equality. Several authors worked on this problem and it is out of the scope of this book to give a list of references. Let us simply mention that the result of Section 3.4.5 can be found in [Vea97b]. Further results on application of tree automata to rigid unification can be found in [DGN$^+$98], [GJV98].

Tree automata are also used in solving classical semantic unification problems. See e.g. [LM93] [KFK97] [Uri92]. For instance, in [KFK97], the idea is to capture some loops in the narrowing procedure using tree automata.

3.6.6 Application of tree automata to decision problems in term rewriting

Some of the applications of tree automata to term rewriting follow from the results on encompassment theory. Early works in this area are also mentioned in the bibliographic notes of Chapter 1. The reader is also referred to the survey [GT95].

The first-order theory of the binary (many-steps) reduction relation w.r.t. a ground rewrite system has been shown decidable by. M. Dauchet and S. Tison [DT90]. Extensions of the theory, including some function symbols, or other predicate symbols like the parallel rewriting or the termination predicate (Terminate(t) holds if there is no infinite reduction sequence starting from t), or fair termination etc... remain decidable [DT90]. **Mauvaise citation !? See also the exercises.

Both the theory of one step and the theory of many steps rewriting are undecidable for arbitrary $\mathcal{R}$ [Tre96].

Reduction strategies for term rewriting have been first studied by Huet and Lévy in 1978 [HL91]. They show here the decidability of *strong sequentiality* for orthogonal rewrite systems. This is based on an approximation of the rewrite system which, roughly, only considers the left sides of the rules. Better approximation, yielding refined criteria were further proposed in [Oya93], [Com95], [Jac96]. The orthogonality requirement has also been replaced with the weaker condition of left linearity. The first relation between tree automata, WSkS and reduction strategies is pointed out in [Com95]. Further studies of call-by-need strategies, which are still based on tree automata, but do not use a detour through monadic second-order logic can be found in [DM97]. For all these works, a key property is the preservation of regularity by (many-steps) rewriting, which was shown for ground systems in [Bra69], for linear systems which do not share variables in [DT90], for shallow systems in [Com95], for right linear monadic rewrite systems [Sal88], for linear semi-monadic rewrite systems [CG90], also called (with slight differences) growing systems in [Jac96]. Growing systems are the currently most general class for which the preservation of recognizability is known.

As already pointed out, the decidability of the encompassment theory implies the decidability of ground reducibility. There are several papers written along these lines which will be explained in the next chapter.

Finally, approximations of the reachable terms are computed in [Gen97] using tree automata techniques, which implies the decision of some safety properties.

3.6.7 Other applications

The relationship between finite tree automata and higher-order matching is studied in [CJ97].

Finite tree automata are also used in logic programming [FSVY91], type reconstruction [Tiu92] and automated deduction [GMW97].

For further applications of tree automata in the direction of program verification, see e.g. chapter 5 of this book or e.g. [Jon87].

Chapter 4

Automata with constraints

4.1 Introduction

A typical example of a language which is not recognized by a finite tree automaton is the set of terms $\{f(t,t) \mid t \in T(\mathcal{F})\}$. The reason is that the two sons of the root are recognized independently and only a fixed finite amount of information can be carried up to the root position, whereas t may be arbitrarily large. Therefore, as seen in the application section of the previous chapter, this imposes some linearity conditions, typically when automata techniques are applied to rewrite systems or to sort constraints. The shift from linear to non linear situations can also be seen as a generalization from tree automata to DAG (directed acyclic graphs) automata. This is the purpose of the present chapter: how is it possible to extend the definitions of tree automata in order to carry over the applications of the previous chapter to (some) non-linear situations ?

Such an extension has been studied in the early 80's by M. Dauchet and J. Mongy. They define a class of automata which (when working in a top-down fashion) allow duplications. Considering bottom-up automata, this amounts to check equalities between subtrees. This yields the *RATEG class* . This class is not closed under complement. If we consider its closure, we get the class of automata with equality and disequality constraints. This class is studied in Section 4.2.

Unfortunately, the emptiness problem is undecidable for the class RATEG (and hence for automata with equality and disequality constraints).
Several decidable subclasses have been studied in the literature. The most remarkable ones are

- The class of *automata with constraints between brothers* which, roughly, allows equality (or disequality) tests only between positions with the same ancestors. For instance, the set of terms $f(t,t)$ is recognized by such an automaton. This class is interesting because all properties of tree automata carry over this extension and hence most of the applications of tree automata can be extended, replacing linearity conditions with such restrictions on non-linearities.

  We study this class in Section 4.3.

- The class of *reduction automata* which, roughly, allows arbitrary disequality constraints but only a fixed finite amount of equality constraints on each run of the automaton. For instance the set of terms $f(t,t)$ also belongs to this class. Though closure properties have to be handled with care (with the definition sketched above, the class is not closed by complement), reduction automata are interesting because for example the set of irreducible terms (w.r.t. an arbitrary, possibly non-linear rewrite system) is recognized by an reduction automaton. Then the decidability of ground reducibility is a direct consequence of emptiness decidability for reduction automata. There is also a logical counterpart: the *reducibility theory* which is presented in the linear case in the previous chapter and which can be shown decidable in the general case using a similar technique.

Reduction automata are studied in Section 4.4.

We also consider in this chapter automata with arithmetic constraints. They naturally appear when some function symbols are assumed to be associative and commutative (AC). In such a situation, the sons of an AC symbol can be permuted and the relevant information is then the number of occurrences of the same subtree in the multisets of sons. These integer variables (number of occurrences) are subject to arithmetic constraints which must belong to a decidable fragment of arithmetic in order to keep closure and decidability properties.

4.2 Automata with equality and disequality constraints

4.2.1 The most general class

An ***equality constraint*** (resp. a ***disequality constraint***) is a predicate on $T(\mathcal{F})$ written $\pi = \pi'$ (resp. $\pi \neq \pi'$) where $\pi, \pi' \in \{1, \ldots, k\}^*$. Such a predicate is satisfied on a term t, which we write $t \models \pi = \pi'$, if $\pi, \pi' \in \mathcal{P}os(t)$ and $t|_\pi = t|_{\pi'}$ (resp. $\pi \neq \pi'$ is satisfied on t if $\pi = \pi'$ is not satisfied on t).

The satisfaction relation $\models$ is extended as usual to any Boolean combination of equality and disequality constraints. The empty conjunction and disjunction are respectively written $\bot$ (false) and $\top$ (true).

An ***automaton with equality and disequality constraints*** is a tuple $(Q, \mathcal{F}, Q_f, \Delta)$ where $\mathcal{F}$ is a finite ranked alphabet, Q is a finite set of states, Q_f is a subset of Q of ***finite states*** and Δ is a set of transition rules of of the form

$$f(q_1, \ldots, q_n) \xrightarrow{c} q$$

where $f \in \mathcal{F}$, $q_1, \ldots, q_n, q \in Q$, and c is a Boolean combination of equality (and disequality) constraints. The state q is called ***target state*** in the above transition rule.

We write for short AWEDC the class of automata with equality and disequality constraints.

Let $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta) \in AWEDC$. The move relation $\rightarrow_\mathcal{A}$ is defined by as for NFTA modulo the satisfaction of equality and disequality constraints: let $t, t' \in F(\mathcal{F} \cup Q, X)$, then $t \rightarrow_\mathcal{A} t'$ if and only

there is a context $C \in \mathcal{C}(\mathcal{F} \cup Q)$ and some terms $u_1, \ldots, u_n \in T(\mathcal{F})$

there exists $f(q_1,\ldots,q_n) \xrightarrow{c} q \in \Delta$

$t = C[f(q_1(u_1),\ldots,q_n(u_n)]$ and $t' = C[q(f(u_1,\ldots,u_n))]$

$C[f(u_1,\ldots,u_n)] \models c$

$\xrightarrow{*}_{\mathcal{A}}$ is the reflexive and transitive closure of $\rightarrow_{\mathcal{A}}$. As in Chapter 1, we usually write $t \xrightarrow{*}_{\mathcal{A}} q$ instead of $t \xrightarrow{*}_{\mathcal{A}} q(t)$.

An automaton $\mathcal{A} \in AWEDC$ ***accepts*** (or ***recognizes***) a ground term $t \in T(\mathcal{F})$ if $t \xrightarrow{*}_{\mathcal{A}} q$ for some state $q \in Q_f$. More generally, we also say that $\mathcal{A}$ accepts t in state q iff $t \xrightarrow{*}_{\mathcal{A}} q$ (acceptance by $\mathcal{A}$ is the particular case of acceptance by $\mathcal{A}$ in a final state).

A ***run***) is a mapping ρ from $\mathcal{P}os(t)$ into Δ such that:

- $\rho(\Lambda) \in Q_f$
- if $t(p) = f$ and the target state of $\rho(p)$ is q, then there is a transition rule $f(q_1,\ldots,q_n) \xrightarrow{c} q$ in Δ such that for all $1 \leq i \leq n$, the target state of $\rho(pi)$ is q_i and $t|_p \models c$.

Note that we do not have here exactly the same definition of a run as in Chapter 1: instead of the state, we keep also the rule which yielded this state. This will be useful in the design of an emptiness decision algorithm for non-deterministic automata with equality and disequality constraints.

The ***language accepted*** (or ***recognized***) by an automaton $\mathcal{A} \in AWEDC$ is the set $L(\mathcal{A})$ of terms $t \in T(\mathcal{F})$ that are accepted by $\mathcal{A}$.

Example 34. Balanced complete binary trees over the alphabet f (binary) and a (constant) are recognized by the AWEDC $(\{q\},\{f,a\},\{q\},\Delta)$ where Δ consists of the following rules:

$$\begin{array}{ll} r_1: & a \rightarrow q \\ r_2: & f(q,q) \xrightarrow{1=2} q \end{array}$$

For example, $t = f(f(a,a),f(a,a))$ is accepted. The mapping which associates r_1 to every position p of t such that $t(p) = a$ and which associates r_2 to every position p of t such that $t(p) = f$ is indeed a successful run: for every position p of t such that $t(p) = f$, $t|_{p\cdot 1} = t_{p\cdot 2}$, hence $t|_p \models 1 = 2$.

Example 35. Consider the following AWEDC: $(Q,\mathcal{F},Q_f,\Delta)$ with $\mathcal{F} = \{0,s,f\}$ where 0 is a constant, s is unary and f has arity 4, $Q = \{q_n,q_0,q_f\}$, $Q_f = \{q_f\}$, and Δ consists of the following rules:

$$\begin{array}{rcl rcl} 0 & \rightarrow & q_0 & s(q_0) & \rightarrow & q_n \\ s(q_n) & \rightarrow & q_n & f(q_0,q_0,q_n,q_n) & \xrightarrow{3=4} & q_f \\ f(q_0,q_0,q_0,q_0) & \rightarrow & q_f & f(q_0,q_n,q_0,q_n) & \xrightarrow{2=4} & q_f \\ f(q_f,q_n,q_n,q_n) & \xrightarrow{14=4\wedge 21=12\wedge 131=3} & q_f & & & \end{array}$$

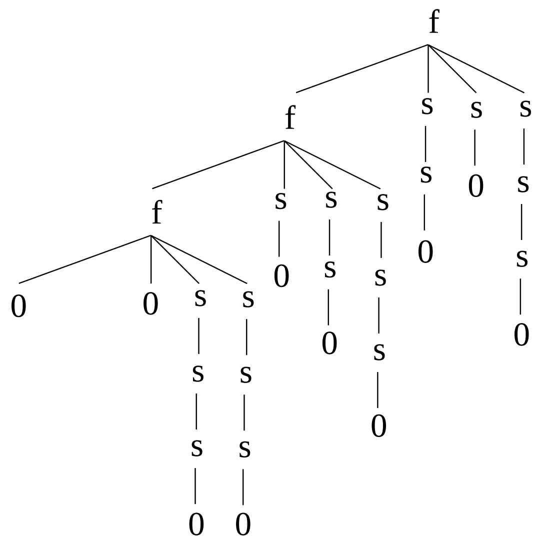


Figure 4.1: A computation of the sum of two natural numbers

This automaton computes the sum of two natural numbers written in base one in the following sense: if t is accepted by $\mathcal{A}$ then[1] $t = f(t_1, s^n(0), s^m(0), s^{n+m}(0))$ for some t_1 and $n, m \geq 0$. Conversely, for each $n, m \geq 0$, there is a term $f(t_1, s^n(0), s^m(0), s^{n+m}(0))$ which is accepted by the automaton.

For instance the term depicted on Figure 4.1 is accepted by the automaton. Similarly, it is possible to design an automaton of the class AWEDC which "computes the multiplication" (see exercises)

In order to evaluate the complexity of operations on automata of the class AWEDC, we need to precise a representation of the automata and estimate the space which is necessary for this representation.

The ***size*** of is a Boolean combination of equality and disequality constraints is defined by induction:

- $\|\pi = \pi'\| \stackrel{\text{def}}{=} \|\pi \neq \pi'\| \stackrel{\text{def}}{=} |\pi| + |\pi'|$ ($|\pi|$ is the length of π)
- $\|c \wedge c'\| \stackrel{\text{def}}{=} \|c \vee c'\| \stackrel{\text{def}}{=} \|c\| + \|c'\| + 1$
- $\|\neg c\| \stackrel{\text{def}}{=} \|c\|$

Now, deciding whether $t \models c$ depends on the representation of t. If t is represented as a directed acyclic graph (a DAG) with maximal sharing, then this can be decided in $O(\|c\|)$ on a RAM. Otherwise, it requires to compute first this

[1] $s^n(0)$ denotes $\underbrace{s(\ldots s}_{n}(0)$

representation of t, and hence can be computed in time at most $O(\|t\| \log \|t\| + \|c\|)$.

¿From now on, we assume, for complexity analysis, that the terms are represented with maximal sharing in such a way that checking an equality or a disequality constraint on t can be completed in a time which is independent of $\|t\|$.

The ***size*** of an automaton $\mathcal{A} \in AWEDC$ is

$$\|\mathcal{A}\| \stackrel{\text{def}}{=} |Q| + \sum_{f(q_1,\dots,q_n) \xrightarrow{c} q \in \Delta} n + 2 + \|c\|$$

An automaton $\mathcal{A}$ in AWEDC is ***deterministic*** if for every $t \in T(\mathcal{F})$, there is at most one state q such that $t \xrightarrow[\mathcal{A}]{*} q$. It is ***complete*** if for every $t \in T(\mathcal{F})$ there is at least one state q such that $t \xrightarrow[\mathcal{A}]{*} q$.

When every constraint is a tautology, then our definition of automata reduces to the definition of Chapter 1. However, in such a case, the notions of determinacy do not fully coincide, as noticed in Chapter 1, page 17.

Proposition 18. *Given $t \in T(\mathcal{F})$ and $\mathcal{A} \in$AWEDC, deciding whether t is accepted by $\mathcal{A}$ can be completed in polynomial time (linear time for a deterministic automaton).*

Proof. Because of the DAG representation of t, the satisfaction of a constraint $\pi = \pi'$ on t can be completed in time $O(|\pi| + |\pi'|)$. Thus, if $\mathcal{A}$ is deterministic, the membership test can be performed in time $O(\|t\| + \|\mathcal{A}\| + MC)$ where $MC = \max(\|c\| \mid c$ is a constraint of a rule of $\mathcal{A})$. If $\mathcal{A}$ is nondeterministic, the complexity of the algorithm will be $O(\|t\| \times \|\mathcal{A}\| \times MC)$. □

4.2.2 Reducing non-determinism and closure properties

Proposition 19. *For every automaton $\mathcal{A} \in AWEDC$, there is a complete automaton $\mathcal{A}'$ which accepts the same language as $\mathcal{A}$. The size $\|\mathcal{A}'\|$ is polynomial in $\|\mathcal{A}\|$ and the computation of $\mathcal{A}'$ can be performed in polynomial time (for a fixed alphabet). If $\mathcal{A}$ is deterministic, then $\mathcal{A}'$ is deterministic.*

Proof. The proof is the same as for Theorem 2: we add a trash state and every transition is possible to the trash state. However, this does not keep the determinism of the automaton. We need the following more careful computation in order to preserve the determinism.

We also add a single trash state $q_\perp$. The additional transitions are computed as follows: for each function symbol $f \in \mathcal{F}$ and each tuple of states (including the trash state) $q_1, \dots, q_n$, if there is no transition $f(q_1, \dots, q_n) \xrightarrow{c} q \in \Delta$, then we simply add the rule $f(q_1, \dots, q_n) \to q_\perp$ to Δ. Otherwise, let $f(q_1, \dots, q_n) \xrightarrow{c_i} s_i$ $(i = 1, ..m)$ be all rules in Δ whose left member is $f(q_1, \dots, q_n)$. We add the rule $f(q_1, \dots, q_n) \xrightarrow{c'} q_\perp$ to Δ, where $c' \stackrel{\text{def}}{=} \neg \bigvee_{i=1}^{m} c_i$. □

Proposition 20. *For every automaton $\mathcal{A} \in AWEDC$, there is a deterministic automaton $\mathcal{A}'$ which accepts the same language as $\mathcal{A}$. $\mathcal{A}'$ can be computed in* *exponential time and its size is exponential in the size of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is complete, then $\mathcal{A}'$ is complete.*

Proof. The construction is the same as in Theorem 4: states of $\mathcal{A}'$ are sets of states of $\mathcal{A}$. Final states of $\mathcal{A}'$ are those which contain at least one final state of $\mathcal{A}$. The construction time complexity as well as the size $\mathcal{A}'$ are also of the same magnitude as in Theorem 4. The only difference is the computation of the constraint: if $S_1, \dots, S_n, S$ are sets of states, in the deterministic automaton, the rule $f(S_1, \dots, S_n) \xrightarrow{c} S$ is labeled by a constraint c defined by:

$$c = \Big(\bigwedge_{q \in S} \bigvee_{\substack{f(q_1,\dots,q_n) \xrightarrow{c_r} q \in \Delta \\ q_i \in S_i i \leq n}} c_r \Big) \wedge \Big(\bigwedge_{q \notin S} \bigwedge_{\substack{f(q_1,\dots,q_n) \xrightarrow{c_r} q \in \Delta \\ q_i \in S_i i \leq n}} \neg c_r \Big)$$

Let us prove that t is accepted by $\mathcal{A}$ in states $q_1, \dots, q_k$ (and no other states) if and only if there t is accepted by $\mathcal{A}'$ in the state $\{q_1, \dots, q_k\}$:

$\Rightarrow$ Assume that $t \xrightarrow[\mathcal{A}]{n} q_i$ (*i.e.* $t \xrightarrow[\mathcal{A}]{*} q_i$ in n steps), for $i = 1, \dots, k$. We prove, by induction on n, that

$$t \xrightarrow[\mathcal{A}']{n} \{q_1, \dots, q_k\}.$$

If $n = 1$, then t is a constant and $t \rightarrow S$ is a rule of $\mathcal{A}'$ where $S = \{q \mid a \xrightarrow[\mathcal{A}]{} q\}$.

Assume now that $n > 1$. Let, for each $i = 1, \dots, k$,

$$t = f(t_1, \dots, t_p) \xrightarrow[\mathcal{A}]{m} f(q_1^i, \dots, q_p^i) \xrightarrow[\mathcal{A}]{} q_i$$

and $f(q_1^i, \dots, q_p^i) \xrightarrow{c_i} q_i$ be a rule of $\mathcal{A}$ such that $t \models c_i$. By induction hypothesis, each term t_j is accepted by $\mathcal{A}'$ in the states of a set $S_j \supseteq \{q_j^1, \dots, q_j^k\}$. Moreover, by definition of $S = \{q_1, \dots, q_k\}$, if $t \xrightarrow[\mathcal{A}]{*} q'$ then $q' \in S$. Therefore, for every transition rule of $\mathcal{A}$ $f(q_1', \dots, q_p') \xrightarrow{c'} q'$ such that $q' \notin S$ and $q_j \in S_j$ for every $j \leq p$, we have $t \not\models c'$. Then t satisfies the above defined constraint c.

$\Leftarrow$ Assume that $t \xrightarrow[\mathcal{A}']{n} S$. We prove by induction on n that, for every $q \in S$, $t \xrightarrow[\mathcal{A}]{n} q$.

If $n = 1$, then S is the set of states q such that $t \xrightarrow[\mathcal{A}]{} q$, hence the property.

Assume now that

$$t = f(t_1, \dots, t_p) \xrightarrow[\mathcal{A}']{n} f(S_1, \dots, S_p) \xrightarrow[\mathcal{A}']{} S.$$

Let $f(S_1, \dots, S_p) \xrightarrow{c} S$ be the last rule used in this reduction. Then $t \models c$ and, by definition of c, for every state $q \in S$, there is a rule $f(q_1, \dots, q_n) \xrightarrow{c_r} q \in \Delta$ such that $q_i \in S_i$ for every $i \leq n$ and $t \models c_r$. By induction hypothesis, for each i, $t_i \xrightarrow[\mathcal{A}']{m_i} S_i$ implies $t_i \xrightarrow[\mathcal{A}]{m_i} q_i$ $(m_i < n)$ and hence $t \xrightarrow[\mathcal{A}]{n} f(q_1, \dots, q_p) \xrightarrow[\mathcal{A}]{} q$.

Thus, by construction of the final states set, a ground term t is accepted by $\mathcal{A}'$ iff t is accepted by $\mathcal{A}$.

Now, we have to prove that $\mathcal{A}'$ is deterministic indeed. Assume that $t \xrightarrow[\mathcal{A}]{*} 'S$ and $t \xrightarrow[\mathcal{A}']{*} S'$. Assume moreover that $S \neq S'$ and that t is the smallest term (in size) with the property of being recognized in two different states. Then there exists $S_1, \dots, S_n$ such that $t \xrightarrow[\mathcal{A}']{*} f(S_1, \dots, S_n)$ and such that $f(S_1, \dots, S_n) \xrightarrow{c} S$ and $f(S_1, \dots, S_n) \xrightarrow{c'} S'$ are transition rules of $\mathcal{A}'$, wit $t \models c$ and $t \models c'$. By symmetry, we may assume that there is a state $q \in S$ such that $q \notin S'$. Then, by definition, there are some states $q_i \in S_i$, for every $i \leq n$, and a rule $f(q_1, \dots, q_n) \xrightarrow{c_r} q$ of $\mathcal{A}$ where c_r occurs positively in c, and is therefore satisfied by t, $t \models c_r$. By construction of the constraint of $\mathcal{A}'$, c_r must occur negatively in the second part of (the conjunction) c'. Therefore, $t \models c'$ contradicts $t \models c_r$. $\square$

Example 36. Consider the following automaton on the alphabet $\mathcal{F} = \{a, f\}$ where a is a constant and f is a binary symbol: $Q = \{q, q_\perp\}$, $Q_f = \{q\}$ and Δ contains the following rules:

$$\begin{array}{ccc} a \to q & f(q,q) \xrightarrow{1=2} q & f(q,q) \to q_\perp \\ f(q_\perp,q) \to q_\perp & f(q,q_\perp) \to q_\perp & f(q_\perp,q_\perp) \to q_\perp \end{array}$$

This is the (non-deterministic) complete version of the automaton of Example 34.

Then the deterministic automaton computed as in the previous proposition is given by:

$$\begin{array}{cc}
a \to \{q\} & f(\{q\},\{q\}) \xrightarrow{1=2\wedge\perp} \{q\} \\
f(\{q\},\{q\}) \xrightarrow{1=2} \{q,q_\perp\} & f(\{q\},\{q\}) \xrightarrow{1\neq2} \{q_\perp\} \\
f(\{q\},\{q_\perp\}) \to \{q_\perp\} & f(\{q_\perp\},\{q\}) \to \{q_\perp\} \\
f(\{q_\perp\},\{q_\perp\}) \to \{q_\perp\} & f(\{q,q_\perp\},\{q\}) \xrightarrow{1=2\wedge\perp} \{q\} \\
f(\{q,q_\perp\},\{q\}) \xrightarrow{1=2} \{q,q_\perp\} & f(\{q,q_\perp\},\{q_\perp\}) \to \{q_\perp\} \\
f(\{q,q_\perp\},\{q,q_\perp\}) \xrightarrow{1=2} \{q,q_\perp\} & f(\{q,q_\perp\},\{q\}) \xrightarrow{1\neq2} \{q_\perp\} \\
f(\{q\},\{q,q_\perp\}) \xrightarrow{1=2} \{q,q_\perp\} & f(\{q\},\{q,q_\perp\}) \xrightarrow{1\neq2} \{q_\perp\} \\
f(\{q,q_\perp\},\{q\}) \xrightarrow{1\neq2} \{q_\perp\} & f(\{q\},\{q,q_\perp\}) \xrightarrow{1=2\wedge\perp} \{q\} \\
f(\{q,q_\perp\},\{q,q_\perp\}) \xrightarrow{1=2\wedge\perp} \{q\} & f(\{q_\perp\},\{q,q_\perp\}) \to \{q_\perp\}
\end{array}$$

For instance, the constraint $1{=}2\wedge\perp$ is obtained by the conjunction of the label of $f(q,q) \xrightarrow{1=2} q$ and the negation of the constraint labelling $f(q,q) \to q_\perp$, (which is $\top$).

Some of the constraints, such as $1{=}2\wedge\perp$ are unsatisfiable, hence the corresponding rules can be removed. If we finally rename the two accepting states $\{q\}$ and $\{q, q_\perp\}$ into a single state q_f (this is possible since by replacing one of these states by the other in any left hand side of a transition rule, we get another transition rule), then we get a simplified version of the deterministic automaton:

$$\begin{array}{rr} a \to q_f & f(q_f, q_f) \xrightarrow{1=2} q_f \\ f(q_f, q_f) \xrightarrow{1\neq 2} q_\perp & f(q_\perp, q_f) \to q_\perp \\ f(q_f, q_\perp) \to q_\perp & f(q_\perp, q_\perp) \to q_\perp \end{array}$$

Proposition 21. *The class AWEDC is effectively closed by all Boolean operations. Union requires linear time, intersection requires quadratic time and complement requires exponential time. The respective sizes of the AWEDC obtained by these construction are of the same magnitude as the time complexity.*

Proof. The proof of this proposition can be obtained from the proof of Theorem 5 (Chapter 1, pages 23–24) with straightforward modifications. The only difference lies in the product automaton for the intersection: we have to consider conjunctions of constraints. More precisely, if we have two AWEDC $\mathcal{A}_1 = (Q_1, \mathcal{F}, Q_{f1}, \Delta_1)$ and $\mathcal{A}_1 = (Q_2, \mathcal{F}, Q_{f2}, \Delta_2)$, we construct an AWEDC $\mathcal{A} = (Q_1 \times Q_2, \mathcal{F}, Q_{f1} \times Q_{f2}, \Delta)$ such that if $f(q_1, \ldots, q_n) \xrightarrow{c} q \in \Delta_1$ and $f(q'_1, \ldots, q'_n) \xrightarrow{c'} q' \in \Delta_2$, then $f((q_1, q'_1), \ldots, (q_n, q'_n)) \xrightarrow{c \wedge c'} (q, q') \in \Delta$. The AWEDC $\mathcal{A}$ recognizes $L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$. $\square$

4.2.3 Undecidability of emptiness

Theorem 29. *The emptiness problem for AWEDC is undecidable.*

Proof. We reduce the Post Correspondence Problem (PCP). If $w_1, \ldots, w_n$ and $w'_1, \ldots, w'_n$ are the word sequences of the PCP problem over the alphabet $\{a, b\}$, we let $\mathcal{F}$ contain h (ternary), a, b (unary) and 0 (constant). Lets recall that the answer for the above instance of the PCP is a sequence $i_1, \ldots, i_p$ (which may contain some repetitions) such that $w_{i_1} \ldots w_{i_p} = w'_{i_1} \ldots w'_{i_p}$.

If $w \in \{a, b\}^*$, $w = a_1 \ldots a_k$ and $t \in T(\mathcal{F})$, we write $w(t)$ the term $a_1(\ldots(a_k(t))\ldots) \in T(\mathcal{F})$.

Now, we construct $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta) \in AWEDC$ as follows:

- Q contains a state q_v for each prefix v of one of the words w_i, w'_i (including q_{w_i} and $q_{w'_i}$ as well as 3 extra states: q_0, q and q_f. We assume that a and b are both prefix of at least one of the words w_i, w'_i. $Q_f = \{q_f\}$.

- Δ contains the following rules:

$$\begin{array}{rcl@{\qquad}rcl} a(q_0) & \to & q_a & b(q_0) & \to & q_b \\ a(q_v) & \to & q_{a\cdot v} & \multicolumn{3}{l}{\text{if } q_v, q_{a\cdot v} \in Q} \\ b(q_v) & \to & q_{b\cdot v} & \multicolumn{3}{l}{\text{if } q_v, q_{b\cdot v} \in Q} \\ a(q_{w_i}) & \to & q_a & b(q_{w_i}) & \to & q_b \\ a(q_{w'_i}) & \to & q_a & b(q_{w'_i}) & \to & q_b \end{array}$$

Δ also contains the rules:

$$\begin{array}{rcl}
0 & \rightarrow & q_0 \\
h(q_0, q_0, q_0) & \rightarrow & q \\
h(q_{w_i}, q, q_{w'_i}) & \xrightarrow{1 \cdot 1^{|w_i|} = 2 \cdot 1 \wedge 3 \cdot 1^{|w'_i|} = 2 \cdot 3} & q \\
h(q_{w_i}, q, q_{w'_i}) & \xrightarrow{1 \cdot 1^{|w_i|} = 2 \cdot 1 \wedge 3 \cdot 1^{|w'_i|} \wedge 1 = 3} & q_f
\end{array}$$

The rule with left member $h(q_0, q_0, q_0)$ recognizes the beginning of a Post sequence. The rules with left members $h(q_{w_i}, q.q_{w'_i})$ ensure that we are really in presence of a successor in the PCP sequences: the constraint expresses that the subterm at position 1 is obtained by concatenating some w_i with the term at position $2 \cdot 1$ and that the subterm at position 3 is obtained by concatenating w'_i (with the same index i) with the subterm at position $2 \cdot 3$. Finally, entering the final state is subject to the additional constraint $1 = 3$. This last constraint expresses that we went thru two identical words with the w_i sequences and the w'_i sequences respectively. (See Figure 4.2).

The details that this automaton indeed accepts the solutions of the PCP are left to the reader.

Then the language accepted by $\mathcal{A}$ is empty if and only if the PCP has a solution. Since PCP is undecidable, emptiness of $\mathcal{A}$ is also undecidable. □

4.3 Automata with constraints between brothers

The undecidability result of the previous section led to look for subclasses which have the desired closure properties, contain (properly) the classical tree automata and still keep the decidability of emptiness. This is the purpose of the class AWCBB:

An automaton $\mathcal{A} \in AWEDC$ is an ***automaton with constraints between brothers*** if every equality (resp disequality) constraint has the form $i = j$ (resp. $i \neq j$) where $i, j \in \mathbb{N}_+$.

AWCBB is the set automata with constraints between brothers.

Example 37. The set of terms $\{f(t, t) \mid t \in T(\mathcal{F})\}$ is accepted by an automaton of the class AWCBB, because the automaton of Example 34 is in AWCBB indeed.

4.3.1 Closure properties

Proposition 22. *AWCBB is a stable subclass of AWEDC w.r.t. Boolean operations (union, intersection, complementation).*

Proof. It is sufficient to check that the constructions of Proposition 21 preserve the property of being a member of AWCBB. □

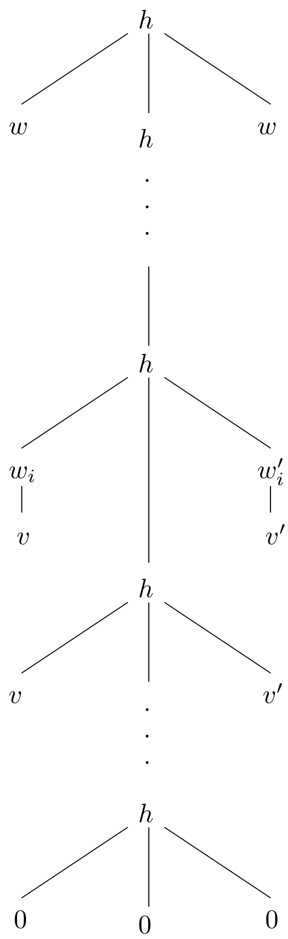


Figure 4.2: An automaton in AWEDC accepting the solutions of PCP

Recall that the time complexity of each such construction is the same in AWEDC and in the unconstrained case: union and intersection are polynomial, complementation requires determinization and is exponential.

4.3.2 Emptiness decision

To decide emptiness we would like to design for instance a "cleaning algorithm" as in Theorem 10. As in this result, the correctness and completeness of the marking technique relies on a pumping lemma. Is there an analog of Lemma 1 in the case of automata of the class AWCBB ?

There are additional difficulties. For instance consider the following example.

Example 38. $\mathcal{A}$ contains only one state and the rules

$$\begin{array}{lcl} a & \rightarrow & q \\ b & \rightarrow & q \end{array} \qquad f(q,q) \xrightarrow{1\neq 2} q$$

Now consider the term $f(f(a,b),b)$ which is accepted by the automaton. $f(a,b)$ and b yield the same state q. Hence, for a classical finite tree automaton, we may replace $f(a,b)$ with b and still get a term which is accepted by $\mathcal{A}$. This is not the case here since, replacing $f(a,b)$ with b we get the term $f(b,b)$ which is not accepted. The reason of this phenomenon is easy to understand: some constraint which was satisfied before the pumping is no longer valid after the pumping.

Hence the problem is to preserve the satisfaction of constraints along term replacements. First, concerning equality constraints, we may see the terms as DAGs in which each pair of subterms which is checked for equality is considered as a single subterm referenced in two different ways. Then replacing one of its occurrences automatically replaces the other occurrences and preserves the equality constraints. This is what is formalized below.

Preserving the equality constraints. Let t be a term accepted by the automaton $\mathcal{A}$ in AWCBB. Let ρ be a run of the automaton on t. With every position p of t, we associate the conjunction $\mathit{cons}(p)$ of atomic (equality or disequality) constraints that are checked by $\rho(p)$ and satisfied by t. More precisely: let $\rho(p) = f(q_1,\dots,q_n) \xrightarrow{c'} q$; $\mathit{cons}(p) \overset{\text{def}}{=} \mathit{decomp}(c',p)$ where $\mathit{decomp}(c',p)$ is recursively defined by: $\mathit{decomp}(\top,p) \overset{\text{def}}{=} \top$, $\mathit{decomp}(c_1 \wedge c_2, p) \overset{\text{def}}{=} \mathit{decomp}(c_1,p) \wedge \mathit{decomp}(c_2,p)$ and $\mathit{decomp}(c_1 \vee c_2, p) = \mathit{decomp}(c_1,p)$ if $t|_p \models c_1$, $\mathit{decomp}(c_1 \vee c_2,p) = \mathit{decomp}(c_2,p)$ otherwise. We can show by a simple induction that $t|_p \models \mathit{cons}(p)$.

Now, we define the equivalence relation $=_t$ on the set of positions of t as the least equivalence relation such that:

- if $i = j \in \mathit{cons}(p)$, then $p \cdot i =_t p \cdot j$
- if $p =_t p'$ and $p \cdot \pi \in \mathcal{P}os(t)$, then $p \cdot \pi =_t p' \cdot \pi$

Note that in the last case, we have $p' \cdot \pi \in \mathcal{P}os(t)$. Of course, if $p =_t p'$, then $t|_p = t|_{p'}$ (but the converse is not necessarily true). Note also (and this is a property of the class AWCBB) that $p =_t p'$ implies that the lengths of p and p' are the same, hence, if $p \neq p'$, they are incomparable w.r.t. the prefix ordering. We can also derive from this remark that each equivalence class is finite (we may assume that each equality constraint of the form $i = i$ has been replaced by $\top$).

Example 39. Consider the automaton whose transition rules are:

$$\begin{array}{rrcl rrcl}
r1: & f(q,q) & \rightarrow & q & r2: & a & \rightarrow & q \\
r3: & f(q,q) & \xrightarrow{1=2} & q_f & r4: & b & \rightarrow & q \\
r5: & f(q,q_f) & \rightarrow & q_f & r6: & f(q_f,q) & \rightarrow & q_f \\
r7: & f(q,q_f) & \rightarrow & q & r8: & f(q_f,q) & \rightarrow & q
\end{array}$$

Let $t = f(b, f(f(f(a,a), f(a,b)), f(f(a,a), f(a,b))))$. A possible run of $\mathcal{A}$ on t is $r5(r4, r3(r1(r1(r2,r2), r1(r2,r5)), r8(r3(r2,r2), r1(r2,r5))))$ Equivalence classes of positions are:

$$\{\Lambda\}, \{1\}, \{2\}, \{21, 22\}, \{211, 221\}, \{212, 222\},$$
$$\{2111, 2211, 2112, 2212\}, \{2121, 2221\}, \{2122, 2222\}$$

Let us recall the principle of pumping, for finite bottom-up tree automata (see Chapter 1). When a ground term $C[C'[t]]$ (C and C' are two contexts) is such that t and $C'[t]$ are accepted in the same state by an NFTA $\mathcal{A}$, then every term $C[C'^n[t]]$ ($n \geq 0$) is accepted by $\mathcal{A}$ in the same state as $C[C'[t]]$. In other words, any $C[C'^n[t]] \in L(\mathcal{A})$ may be reduced by pumping it up to the term $C[t] \in L(\mathcal{A})$. We consider here a position p (corresponding to the term $C'[t]$) and its equivalence class $[\![p]\!]$ modulo $=_t$. The simultaneous replacement on $[\![p]\!]$ with t in u, written $u[t]_{[\![p]\!]}$, is defined as the term obtained by successively replacing the subterm at position p' with t for each position $p' \in [\![p]\!]$. Since any two positions in $[\![p]\!]$ are incomparable, the replacement order is irrelevant. Now, a ***pumping*** is a pair $(C[C'[t]]_p, C[C'^n[t]]_{[\![p]\!]})$ where $C'[t]$ and t are accepted in the same state.

Preserving the disequality constraints. We have seen on Example 38 that, if t is accepted by the automaton, replacing one of its subterms, say u, with a term v accepted in the same state as u, does not necessary yield an accepted term. However, the idea is now that, if we have sufficiently many such candidates v, at least one of the replacements will keep the satisfaction of disequality constraints.

This is the what shows the following lemma which states that minimal accepted terms cannot contain too many subterms accepted in the same state.

Lemma 5. *Given any total simplification ordering, a minimal term accepted by a deterministic automaton in AWCBB contains at most $|Q| \times N$ distinct subterms where N is the maximal arity of a function symbol and $|Q|$ is the number of states of the automaton.*

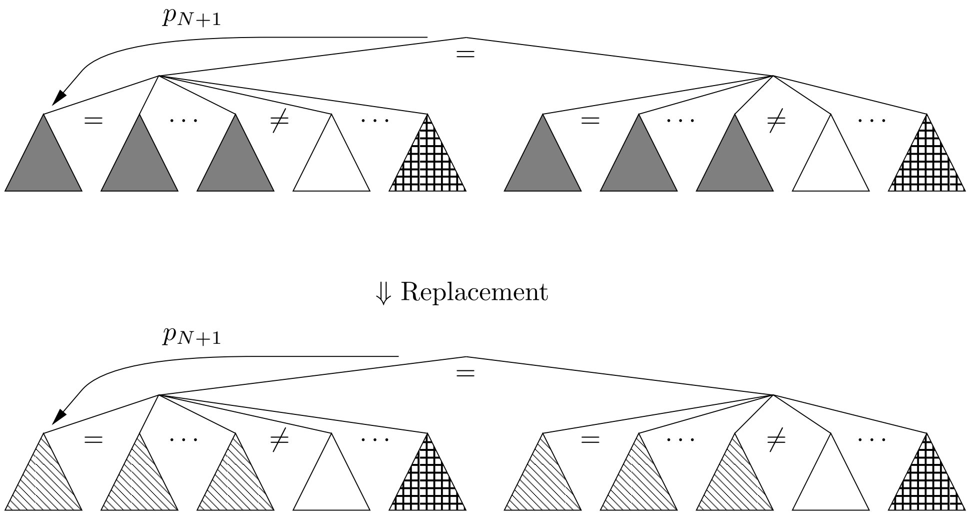


Figure 4.3: Constructing a smaller term accepted by the automaton

Proof. If ρ is a run, let $\tau\rho$ be the mapping from positions to states such that $\tau\rho(p)$ is the target state of $\rho(p)$.

If t is accepted by the automaton (let ρ be a successful run on t) and contains at least $1 + |Q| \times N$ distinct subterms, then there are at least $N + 1$ positions $p_1, \ldots, p_{N+1}$ such that $\tau\rho(p_1) = \ldots = \tau\rho(p_{N+1})$ and $t|_{p_1}, \ldots, t|_{p_{N+1}}$ are distinct. Assume for instance that $t|_{p_{N+1}}$ is the largest term (in the given total simplification ordering) among $t|_{p_1}, \ldots, t|_{p_{N+1}}$. We claim that one of the terms $v_i \overset{\text{def}}{=} t[t|_{p_i}]_{[\![p_{N+1}]\!]}$ $(i \leq N)$ is accepted by the automaton.

For each $i \leq N$, we may define unambiguously a tree ρ_i by: $\rho_i = \rho[\rho|_{p_i}]_{[\![p_{N+1}]\!]}$.

First, note that, by determinacy, for each position $p \in [\![p_{N+1}]\!]$, $\tau\rho(p) = \tau\rho(p_{N+1}) = \tau\rho(p_i)$. To show that there is a ρ_i which is a run, it remains to find a ρ_i the constraints of which are satisfied. Equality constraints of any ρ_i are satisfied, from the construction of the equivalence classes (details are left to the reader).

Concerning disequality constraints, we choose i in such a way that all subterms at brother positions of p_{N+1} are distinct from $t|_{p_i}$ (this choice is possible since N is the maximal arity of a function symbol and there are N distinct candidates). We get a replacement as depicted on Figure 4.3.

Let $p_{N+1} = \pi \cdot k$ where $k \in \mathbb{N}$ (π is the position immediately above p_{N+1}). Every disequality in $cons(\pi)$ is satisfied by choice of i. Moreover, if $p' \in [\![p_{N+1}]\!]$ and $p' = \pi' \cdot k'$ with $k' \in \mathbb{N}$, then every disequality in $mathitcons(\pi')$ is satisfied since $v_i|_\pi = v_i|_{\pi'}$.

Hence we constructed a term which is smaller than t and which is accepted by the automaton. This yields the lemma. $\square$

Theorem 30. *Emptiness can be decided in polynomial time for deterministic automata in AWCBB.*

Proof. The basic idea is that, if we have enough distinct terms in states $q_1, \ldots, q_n$, then the transition $f(q_1, \ldots, q_n) \xrightarrow{c} q$ is possible. Use a marking algorithm (as in Theorem 3) and keep for each state the terms that are known to be accepted in this state. It is sufficient to keep at most N terms in each state (N is the maximal arity of a function symbol) according to Lemma 5 and the determinacy hypothesis (terms in different states are distinct). More precisely, we use the following algorithm:

```
input: AWCBB $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$
begin
    – Marked is a mapping which associates each state with a set of
    terms accepted in that state.
        Set Marked to the function which maps each state to the $\emptyset$
        repeat
            Set $Marked(q)$ to $Marked(q) \cup \{t\}$
            where
                $f \in \mathcal{F}_n$, $t_1 \in Marked(q_1), \dots, t_n \in Marked(q_n)$,
                $f(q_1, \dots, q_n) \xrightarrow{c} q \in \Delta$,
                $t = f(t_1, \dots, t_n)$ and $t \models c$,
                $|Marked(q)| \leq N - 1$,
        until no term can be added to any $Marked(q)$
        output: true if, for every state $q_f \in Q_f$, $Marked(q_f) = \emptyset$.
end
```

□

Complexity. For non-deterministic automata, an exponential time algorithm is derived from Proposition 20 and Theorem 30. Actually, in the non-deterministic case, the problem is EXPTIME-complete.

We may indeed reduce the following problem which is known to be EXPTIME-complete to non-emptiness decision for nondeterministic AWCBB.
Instance n tree automata $\mathcal{A}_1, \dots ,\mathcal{A}_n$ over $\mathcal{F}$.
Answer "yes" iff the intersection the respective languages recognized by $\mathcal{A}_1, \dots ,\mathcal{A}_n$ is not empty.

We may assume without loss of generality that the states sets of $\mathcal{A}_1, \dots ,\mathcal{A}_n$ (called respectively $Q_1, \dots ,Q_n$) are pairwise disjoint, and that every $\mathcal{A}_i$ has a single final state called q_i^f. We also assume that $n = 2^k$ for some integer k. If this is not the case, let k be the smallest integer i such that $n < 2^i$ and let $n' = 2^k$. We consider a second instance of the above problem: $\mathcal{A}'_1, \dots ,\mathcal{A}'_{n'}$ where

$$\mathcal{A}'_i = \mathcal{A}_i \text{ for each } i \leq n.$$

$$\mathcal{A}'_i = (\{q\}, \mathcal{F}, \{q\}, \{f(q, \dots , q) \rightarrow q | f \in \mathcal{F}\}) \text{ for each } n < i \leq n'.$$

Note that the tree automaton in the second case is universal, i.e. it accepts every term of $T(\mathcal{F})$. Hence, the answer is "yes" for $\mathcal{A}'_1, \dots ,\mathcal{A}'_{n'}$ iff it is "yes" for $\mathcal{A}_1, \dots ,\mathcal{A}_n$.

Now, we add a single new binary symbol g to $\mathcal{F}$, getting $\mathcal{F}'$, and consider the following AWCBB $\mathcal{A}$:

$$\mathcal{A} = (\bigcup_{i=1}^{n} Q_i \uplus \{q_1, \dots, q_{n-1}\}, \mathcal{F}', \{q_1\}, \Delta)$$

where $q_1, \dots, q_{2n-1}$ are new states, and the transition of Δ are:

every transition rule of $\mathcal{A}_1, \dots, \mathcal{A}_n$ is a transition rule of $\mathcal{A}$,

for each $i < \frac{n}{2}$, $g(q_{2i}, q_{2i+1}) \xrightarrow{1=2} q_i$ is a transition rule of $\mathcal{A}$,

for each i, $\frac{n}{2} \leq i < n-1$, $g(q^f_{2i}, q^f_{2i+1}) \xrightarrow{1=2} q_i$ is a transition rule of $\mathcal{A}$.

Note that $\mathcal{A}$ is non-deterministic, even if every $\mathcal{A}_i$ is deterministic.

We can show by induction on k ($n = 2^k$) that the answer to the above problem is "yes" iff the language recognized by $\mathcal{A}$ is not empty. Moreover, the size of $\mathcal{A}$ si linear in the size of the initial problem and $\mathcal{A}$ is constructed in a time which is linear in his size. This proves the EXPTIME-hardness of emptiness decision for AWCBB.

4.3.3 Applications

The main difference between AWCBB and NFTA is the non-closure of AWCBB under projection and cylindrification. Actually, the shift from automata on trees to automata on tuples of trees cannot be extended to the class AWCBB.

As long as we are interested in automata recognizing sets of trees, all results on NFTA (and all applications) can be extended to the class AWCBB (with an bigger complexity). For instance, Theorem 23 (sort constraints) can be extended to interpretations of sorts as languages accepted by AWCBB. The Proposition 15 (encompassment) can be easily generalized to the case of non-linear terms in which non-linearities only occur between brother positions, provided that we replace NFTA with AWCBB. Theorem 24 can also be generalized to the reducibility theory with predicates $\trianglelefteq_t$ where t is non-linear terms, provided that non-linearities in t only occur between brother positions.

However, we can no longer invoke an embedding into WSkS. The important point is that this theory only requires the weak notion of recognizability on tuples ($Rec_\times$). Hence we do not need automata on tuples, but only tuples of automata. As an example of application, we get a decision algorithm for ground reducibility of a term t w.r.t. left hand sides $l_1, \dots, l_n$, provided that all non-linearities in $t, l_1, \dots, l_n$ occur at brother positions: simply compute the automata $\mathcal{A}_i$ accepting the terms that encompass l_i and check that $L(\mathcal{A}) \subseteq L(\mathcal{A}_1) \cup \dots \cup L(\mathcal{A}_n)$.

Finally, the application on reduction strategies does not carry over the case of non-linear terms because there really need automata on tuples.

4.4 Reduction automata

As we have seen above, the first-order theory of finitely many unary encompassment predicates $\trianglelefteq_{t_1}, \dots, \trianglelefteq_{t_n}$ (reducibility theory) is decidable when non-linearities in the terms t_i are restricted to brother positions. What happens when we drop the restrictions and consider arbitrary terms $t_1, \dots, t_n, t$? It turns out that the theory remains decidable, as we will see. Intuitively, we make impossible counter examples like the one in the proof of Theorem 29 (stating undecidability of the emptiness problem for AWEDC) with an additional condition that using the automaton which accepts the set of terms encompassing t, we may only check for a bounded number of equalities along each branch. That is the idea of the next definitions of reduction automata.

4.4.1 Definition and closure properties

A ***reduction automaton*** $\mathcal{A}$ is a member of AWEDC such that there is an ordering on the states of $\mathcal{A}$ such that,

> for each rule $f(q_1, \dots, q_n) \xrightarrow{\pi_1=\pi_2 \wedge c} q$, q is strictly smaller than each q_i.

In case of an automaton with ϵ-transitions $q \rightarrow q'$ we also require q' to be not larger than q.

Example 40. Consider the set of terms on the alphabet $\mathcal{F} = \{a, g\}$ encompassing $g(g(x, y), x)$. It is accepted by the following reduction automaton, the final state of which is q_f and q_f is minimal in the ordering on states.

$$\begin{array}{rcl rcl}
a & \rightarrow & q_\top & g(q_\top, q_\top) & \rightarrow & q_{g(x,y)} \\
g(q_\top, q_{g(x,y)}) & \rightarrow & q_{g(x,y)} & & & \\
g(q_{g(x,y)}, q_\top) & \xrightarrow{11=2} & q_f & g(q_{g(x,y)}, q_\top) & \xrightarrow{11\neq 2} & q_{g(x,y)} \\
g(q_{g(x,y)}, q_{g(x,y)}) & \xrightarrow{11=2} & q_f & g(q_{g(x,y)}, q_{g(x,y)}) & \xrightarrow{11\neq 2} & q_{g(x,y)} \\
g(q, q_f) & \rightarrow & q_f & g(q_f, q) & \rightarrow & q_f
\end{array}$$

where $q \in \{q_\top, q_{g(x,y)}, q_f\}$

This construction can be generalized, along the lines of the proof of Proposition 15 (page 86):

Proposition 23. *The set of terms encompassing a term t is accepted by a deterministic and complete reduction automaton. The size of this automaton is polynomial in $\|t\|$ as well as the time complexity for its construction.*

As usual, we are now interested in closure properties:

Proposition 24. *The class of reduction automata is closed under union and intersection. It is not closed under complementation.*

Proof. The constructions for union and intersection are the same as in the proof of Proposition 21, and therefore, the respective time complexity and sizes are the same. The proof that the class of reduction automata is closed under these constructions is left as an exercise. Consider the set L of ground terms on the alphabet $\{a, f\}$ defined by $a \in L$ and for every $t \in L$ which is not a, t has a subterm of the form $f(s, s')$ where $s \neq s'$. The set L is accepted by a (non-deterministic, non-complete) reduction automaton, but its complement is the set of balanced binary trees and it cannot be accepted by a reduction automaton (see Exercise 57). $\square$

The weak point is of course the non-closure under complement. Consequently, this is not possible to reduce the non-determinism.

However, we have a weak version of stability:

Proposition 25.

- *With each reduction automaton, we can associate a complete reduction automaton which accepts the same language. Moreover, this construction preserves the determinism.*
- *The class of complete deterministic reduction automata is closed under complement.*

4.4.2 Emptiness decision

Theorem 31. *Emptiness is decidable for the class of reduction automata.*

The proof of this result is quite complicated and gives quite high upper bounds on the complexity (a tower of several exponentials). Hence, we are not going to reproduce it here. Let us only sketch how it works, in the case of deterministic reduction automata.

As in Section 4.3.2, we have both to preserve equality and disequality constraints.

Concerning equality constraints, we also define an equivalence relation between positions (of equal subtrees). We cannot claim any longer that two equivalent positions do have the same length. However, some of the properties of the equivalence classes are preserved: first, they are all finite and their cardinal can be bounded by a number which only depends on the automaton, because of the condition with the ordering on states (this is actually not true for the class AWCBB). Then, we can compute a bound b_2 (which only depends on the automaton) such that the difference of the lengths of two equivalent positions is smaller than b_2. Nevertheless, as in Section 4.3.2, equalities are not a real problem, as soon as the automaton is deterministic. Indeed, pumping can then be defined on equivalence classes of positions. If the automaton is not deterministic, the problem is more difficult since we cannot guarantee that we reach the same state at two equivalent positions, hence we have to restrict our attention to some particular runs of the automaton.

Handling disequalities requires more care; the number of distinct subterms of a minimal accepted term cannot be bounded as for AWCBB by $|Q| \times N$, where N is the maximal arity of a function symbol. The problem is the possible "overlap" of disequalities checked by the automaton. As in Example 38, a pumping may yield a term which is no longer accepted, since a disequality checked somewhere in the term is no longer satisfied. In such a case, we say that the pumping *creates an equality*. Then, we distinguish two kinds of equalities created by a pumping: the ***close equalities*** and the ***remote equalities***. Roughly, an equality created by a pumping $(t[v(u)]_p, t[u]_p)$ is a pair of positions $(\pi \cdot \pi_1, \pi \cdot \pi_2)$ of $t[v(u)]_p$ which was checked for disequality by the run ρ at position π on $t[v(u)]_p$ and such that $t[u]_p|_{\pi \cdot \pi_1} = t[u]_p|_{\pi \cdot \pi_2}$ (π is the longest common prefix to both members of the pair). This equality $(\pi \cdot \pi_1, \pi \cdot \pi_2)$ is a close equality if $\pi \leq p < \pi \cdot \pi_1$ or $\pi \leq p < \pi \cdot \pi_2$. Otherwise ($p \geq \pi \cdot \pi_1$ or $p \geq \pi \cdot \pi_2$), it is a remote equality. The different situations are depicted on Figures 4.4 and 4.5.

One possible proof sketch is

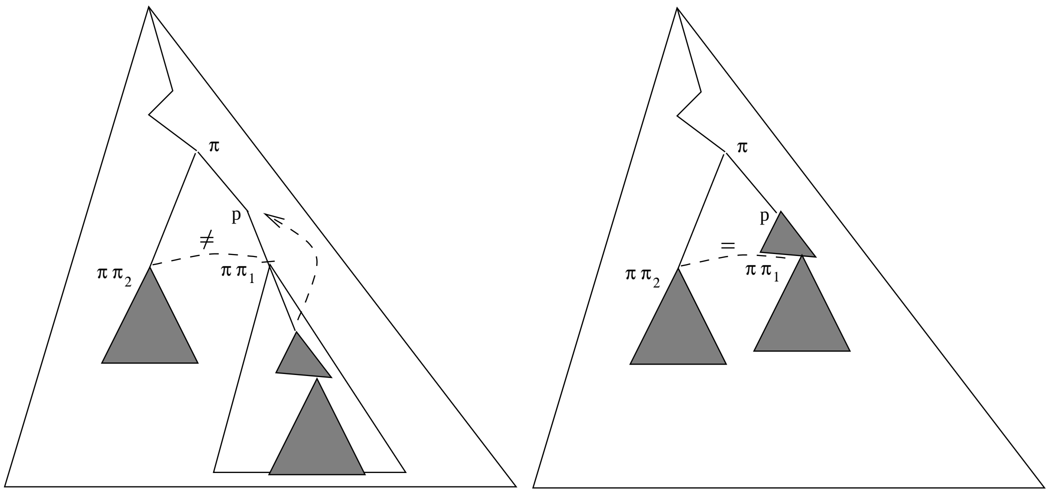


Figure 4.4: A close equality is created

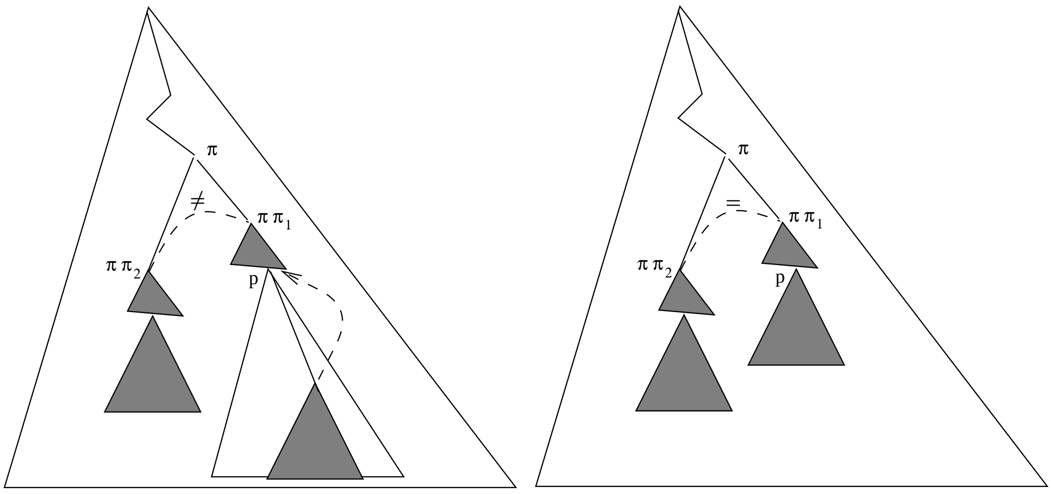


Figure 4.5: A remote equality is created

- First show that it is sufficient to consider equalities that are created at positions around which the states are incomparable w.r.t. $>$
- Next, show that, for a deep enough path, there is at least one pumping which does not yield a close equality (this makes use of a combinatorial argument; the bound is an exponential in the maximal size of a constraint).
- For remote equalities, pumping is not sufficient. However, if some pumping creates a remote equality anyway, this means that there are "big"equal terms in t. Then we switch to another branch of the tree, combining pumping in both subtrees to find one (again using a combinatorial argument) such that no equality is created.

Of course, this is a very sketchy proof. The reader is referred to the bibliography for more information about the proof.

4.4.3 Finiteness decision

The following result is quite difficult to establish. We only mention them for sake of completeness.

Theorem 32. *Finiteness of the language is decidable for the class of reduction automata.*

4.4.4 Term rewriting systems

There is a strong relationship between reduction automata and term rewriting. We mention them readers interested in that topic.

Proposition 26. *Given a term rewriting system $\mathcal{R}$, the set of ground $\mathcal{R}$-normal forms is recognizable by a reduction automaton, the size of which is exponential in the size of $\mathcal{R}$. The time complexity of the construction is exponential.*

Proof. The set of $\mathcal{R}$-reducible ground terms can be defined as the union of sets of ground terms encompassing the left members of rules of $\mathcal{R}$. Thus, by Propositions 23 and 24 the set of $\mathcal{R}$-reducible ground terms is accepted by a deterministic and complete reduction automaton. For the union, we use the product construction, preserving determinism (see the proof of Theorem 5, Chapter 1) with the price of an exponential blowup. The set of ground $\mathcal{R}$-normal forms is the complement of the set of ground $\mathcal{R}$-reducible terms, and it is therefore accepted by a reduction automaton, according to Proposition 25. □

Thus, we have the following consequence of Theorems 32 and 31.

Corollary 5. *Emptiness and finiteness of the language of ground $\mathcal{R}$-normal forms is decidable for every term rewriting system $\mathcal{R}$.*

Let us cite another important result concerning recognizability of sets normal forms.

Theorem 33. *The membership of the language of ground normal forms to the class of recognizable tree languages is decidable.*

4.4.5 Application to the reducibility theory

Consider the reducibility theory of Section 3.4.2: there are unary predicate symbols $\trianglelefteq_t$ which are interpreted as the set of terms which encompass t. However, we accept now non linear terms t as indices.

Propositions 23, and 24, and 25 yield the following result:

Theorem 34. *The reducibility theory associated with any sets of terms is decidable.*

And, as in the previous chapter, we have, as an immediate corollary:

Corollary 6. *Ground reducibility is decidable.*

4.5 Other decidable subclasses

Complexity issues and restricted classes. There are two classes of automata with equality and disequality constraints for which tighter complexity results are known:

- For the class of automata containing only disequality constraints, emptiness can be decided in deterministic exponential time. For any term rewriting system $\mathcal{R}$, the set of ground $\mathcal{R}$-normal forms is still recognizable by an automaton of this subclass of reduction automata.
- For the class of deterministic reduction automata for which the constraints "cannot overlap", emptiness can be decided in polynomial time.

Combination of AWCBB and reduction automata. If you relax the condition on equality constraints in the transition rules of reduction automata so as to allow constraints between brothers, you obtain the biggest known subclass of AWEDC with a decidable emptiness problem.

Formally, these automata, called ***generalized reduction automata***, are members of AWEDC such that there is an ordering on the states set such that,

for each rule $f(q_1, \dots, q_n) \xrightarrow{\pi_1=\pi_2 \wedge c} q$, q is a lower bound of $\{q_1, \dots, q_n\}$ and moreover, if $|\pi_1| > 1$ or $|\pi_2| > 1$, then q is strictly smaller than each q_i.

The closure and decidability results for reduction automata may be transposed to generalized reduction automata, with though a longer proof for the emptiness decision. Generalized reduction automata can thus be used for the decision of reducibility theory extended by some restricted sort declarations. In this extension, additionally to encompassment predicates $\trianglelefteq_t$, we allow a family of unary sort predicates $. \in S$, where S is a sort symbol. But, sort declarations are limited to atoms of the form $t \in S$ where where non linear variables in t only occur at brother positions. This fragment is decidable by an analog of Theorem 34 for generalized reduction automata.

4.6 Tree Automata with Arithmetic Constraints

Tree automata deal with finite trees which have a width bounded by the maximal arity of the signature but there is no limitation on the depth of the trees. A natural idea is to relax the restriction on the width of terms by allowing function of variadic arity. This has been considered by several authors for applications to graph theory, typing in object-oriented languages, temporal logic and automated deduction. In these applications, variadic functions are set or multiset constructors in some sense, therefore they enjoy additional properties like associativity and/or commutativity and several types of tree automata have been designed for handling these properties. We describe here a class of tree automata which recognize terms build with usual function symbols and multiset constructors. Therefore, we deal not only with terms, but with so-called flat terms. Equality on these terms is no longer the syntactical identity, but it is extended by the equality of multisets under permutation of their elements. To recognize sets of flat terms with automata, we shall use constrained rules where the constraints are Presburger's arithmetic formulas which set conditions on the multiplicities of terms in multisets. These automata enjoy similar properties to NFTA and are used to test completeness of function definitions and inductive reducibility when associative-commutative functions are involved, provided that some syntactical restrictions hold.

4.6.1 Flat Trees

The set of function symbols $\mathcal{G}$ is composed of $\mathcal{F}$, the set of function symbols and of $\mathcal{M}$, the set of function symbols for building multisets. For simplicity we shall assume that there is only one symbol of the latter form, denoted by $\sqcup$ and written as an infix operator. The set of variables is denoted by $\mathcal{X}$. ***Flat terms*** are terms generated by the non-terminal T of the following grammar.

$$\begin{array}{rcll}
N & ::= & 1\,|\,2\,|\,3\ldots & \\
T & ::= & S\,|\,U & \text{(flat terms)} \\
S & ::= & x\,|\,f(T_1,\ldots,T_n) & \text{(flat terms of sort } \mathcal{F}\text{)} \\
U & ::= & N_1.S_1 \sqcup \ldots \sqcup N_p.S_p & \text{(flat terms of sort } \sqcup\text{)}
\end{array}$$

where $x \in \mathcal{X}$, $n \geq 0$ is the arity of f, $p \geq 1$ and $\sum_{i=1}^{i=p} N_i \geq 2$. Moreover the inequality $t_i \neq_P t_j$ holds for $i \neq j$, $1 \leq i,j < n$, where $=_P$ is defined as the smallest congruence such that:

- $x =_P x$,
- $f(s_1,\ldots,s_n) =_P f(t_1,\ldots,t_n)$ if $f \in \mathcal{F}$ and $s_i =_P t_i$ for $i = 1,\ldots,n$,
- $n_1.s_1 \sqcup \ldots \sqcup n_p.s_p =_P m_1.t_1 \sqcup \ldots \sqcup m_q.t_q$ if $p = q$ and there is some permutation σ on $\{1,\ldots,p\}$ such that $s_i =_P t_{\sigma(i)}$ and $n_i = m_{\sigma(i)}$ for $i = 1,\ldots,p$.

Example 41. $3.a$ and $3.a \sqcup 2.f(x,b)$ are flat terms, but $2.a \sqcup 1.a \sqcup f(x,b)$ is not since $2.a$ and $1.a$ must be grouped together to make $3.a$.

The usual notions on terms can be generalized easily for flat terms. We recall only what is needed in the following. A flat term is ***ground*** if it contains no variables. The ***root*** of a flat term is defined by

- for the flat terms of sort $\mathcal{F}$, $root(x) = x$, $root(f(t_1, \dots, t_n)) = f$,
- for the flat terms of sort $\sqcup$, $root(s_1 \sqcup \dots \sqcup s_n) = \sqcup$.

Our notion of subterm is slightly different from the usual one. We say that s is a subterm of t if and only if

- either s and t are identical,
- or $t = f(s_1, \dots, s_n)$ and s is a subterm of some s_i,
- or $t = n_1.t_1 \sqcup \dots \sqcup n_p.t_p$ and s is a subterm of some t_i.

For simplicity, we extend $\sqcup$ to an operation between flat terms s, t denoting (any) flat term obtained by regrouping elements of sort $\mathcal{F}$ in s and t which are equivalent modulo $=_P$, leaving the other elements unchanged. For instance $s = 2.a \sqcup 1.f(a, a)$ and $t = 3.b \sqcup 2.f(a, a)$ yields $s \sqcup t = 2.a \sqcup 3.b \sqcup 3.f(a, a)$.

4.6.2 Automata with Arithmetic Constraints

There is some regularity in flat terms that is likely to be captured by some class of automata-like recognizers. For instance, the set of flat terms such that all integer coefficients occurring in the terms are even, seems to be easily recognizable, since the predicate $even(n)$ can be easily decided. The class of automata that we describe now has been designed for accepting such sets of ground flat terms. A ***flat tree automaton with arithmetic constraints*** (NFTAC) over $\mathcal{G}$ is a tuple $(Q_\mathcal{F}, Q_\sqcup, \mathcal{G}, Q_f, \Delta)$ where

- $Q_\mathcal{F} \cup Q_\sqcup$ is a finite set of states, such that
  - $Q_\mathcal{F}$ is the set of states of sort $\mathcal{F}$,
  - $Q_\sqcup$ is the set of states of sort $\sqcup$,
  - $Q_\mathcal{F} \cap Q_\sqcup = \emptyset$,
- $Q_f \subseteq Q_\mathcal{F} \sqcup Q_\sqcup$ is the set of final states,
- Δ is a set of rules of the form:
  - $f(q_1, \dots, q_n) \to q$, for $n \geq 0, f \in \mathcal{F}_n$, $q_1, \dots, q_n \in Q_\mathcal{F} \cup Q_\sqcup$, and $q \in Q_\mathcal{F}$,
  - $N.q \xrightarrow{c(N)} q'$, where $q \in Q_\mathcal{F}$, $q' \in Q_\sqcup$, and c is a Presburger's arithmetic[2] formula with the unique free variable N,
  - $q_1 \sqcup q_2 \to q_3$ where $q_1, q_2, q_3 \in Q_\sqcup$.

Moreover we require that

[2]Presburger's arithmetic is first order arithmetic with addition and constants 0 and 1. This fragment of arithmetic is known to be decidable.

- $q_1 \sqcup q_2 \rightarrow q_3$ is a rule of Δ implies that $q_2 \sqcup q_1 \rightarrow q_3$ is also a rule of Δ,
- $q_1 \sqcup (q_2 \sqcup q_3) \rightarrow q_4$ is a rule of Δ implies that $(q_1 \sqcup q_2) \sqcup q_3 \rightarrow q_4$ is also a rule of Δ where $q_2 \sqcup q_3$ (resp. $q_1 \sqcup q_2$) denotes any state q' such that there is a rule $q_2 \sqcup q_3 \rightarrow q'$ (resp. $q_1 \sqcup q_2 \rightarrow q'$).

These two conditions on Δ will ensure that two flat terms equivalent modulo $=_P$ reach the same states.

Example 42. Let $\mathcal{F} = \{a, f\}$ and let $\mathcal{A} = (Q_{\mathcal{F}}, Q_{\sqcup}, \mathcal{G}, Q_f, \Delta)$ where

$Q_{\mathcal{F}} = \{q\}, Q_{\sqcup} = \{q_{\sqcup}\}$,

$Q_f = \{q_u\}$,

$$\Delta = \left\{ \begin{array}{ccccccc} a & \longrightarrow & q & & N.q & \overset{\exists n : N = 2n}{\longrightarrow} & q_{\sqcup} \\ f(\_,\_) & \longrightarrow & q & & q_{\sqcup} \sqcup q_{\sqcup} & \longrightarrow & q_{\sqcup} \end{array} \right\}$$

where _ stands for q or $q_{\sqcup}$.

Let $\mathcal{A} = (Q_{\mathcal{F}}, Q_{\sqcup}, \mathcal{G}, Q_f, \Delta)$ be a flat tree automaton. The move relation $\rightarrow_{\mathcal{A}}$ is defined by: let $t, t' \in \mathcal{T}(\mathcal{F} \cup Q, X)$, then $t \rightarrow_{\mathcal{A}} t'$ if and only if there is a context $C \in \mathcal{C}(\mathcal{G} \cup Q)$ such that $t = C[s]$ and $t' =_P C[s']$ where

- either there is some $f(q_1, \ldots, q_n) \rightarrow q' \in \Delta$ and $s = f(q_1, \ldots, q_n)$, $s' = q'$,
- or there is some $N.q \xrightarrow{c(N)} q' \in \Delta$ and $s = n.q$ with $\models c(n)$, $s' = q'$,
- or there is some $q_1 \sqcup q_2 \rightarrow q_3 \in \Delta$ and $s = q_1 \sqcup q_2$, $s' = q_3$.

$\xrightarrow{*}_{\mathcal{A}}$ is the reflexive and transitive closure of $\rightarrow_{\mathcal{A}}$.

Example 43. Using the automaton of the previous example, one has

$$\begin{array}{lll} 2.a \sqcup 6.f(a,a) \sqcup 2.f(a, 2.a) & \xrightarrow{*}_{\mathcal{A}} & 2.q \sqcup 6.f(q,q) \sqcup 2.f(q, 2.q) \\ & \xrightarrow{*}_{\mathcal{A}} & 2.q \sqcup 6.q \sqcup 2.f(q, q_{\sqcup}) \\ & \xrightarrow{*}_{\mathcal{A}} & 2.q \sqcup 6.q \sqcup 2.q \\ & \xrightarrow{*}_{\mathcal{A}} & q_{\sqcup} \sqcup q_{\sqcup} \sqcup q_{\sqcup} \quad \xrightarrow{*}_{\mathcal{A}} \quad q_{\sqcup} \sqcup q_{\sqcup} \quad \xrightarrow{*}_{\mathcal{A}} \quad q_{\sqcup} \end{array}$$

We define now ***semilinear flat languages***. Let $\mathcal{A} = (Q_{\mathcal{F}}, Q_{\sqcup}, \mathcal{G}, Q_f, \Delta)$ be a flat tree automaton. A ground flat term t is ***accepted*** by $\mathcal{A}$, if there is some $q \in Q_f$ such that $t \xrightarrow{*}_{\mathcal{A}} q$. The flat tree language $L(\mathcal{A})$ accepted by $\mathcal{A}$ is the set of all ground flat terms accepted by $\mathcal{A}$. A set of flat terms is ***semilinear*** if there $L = L(\mathcal{A})$ for some NFTAC $\mathcal{A}$. Two flat tree automata are ***equivalent*** if they recognize the same language.

Example 44. The language of terms accepted by the automaton of Example 42 is the set of ground flat terms with root $\sqcup$ such that for each subterm $n_1.s_1 \sqcup \ldots \sqcup n_p.s_p$ we have that n_i is an even number.

Flat tree automata are designed to take into account the $=_P$ relation, which is stated by the next proposition.

Proposition 27. *Let s, t, be two flat terms such that $s =_P t$, let $\mathcal{A}$ be a flat tree automaton, then $s \xrightarrow{*}_{\mathcal{A}} q$ implies $t \xrightarrow{*}_{\mathcal{A}} q$.*

Proof. The proof is by structural induction on s. □

Proposition 28. *Given a flat term t and a flat tree automaton $\mathcal{A}$, it is decidable whether t is accepted by $\mathcal{A}$.*

Proof. The decision algorithm for membership for flat tree automata is nearly the same as the one for tree automata presented in Chapter 1, using an oracle for the decision of Presburger's arithmetic formulas. □

Our definition of flat tree automata corresponds to nondeterministic flat tree automata. We now define deterministic flat tree automata (DFTAC).
Let $\mathcal{A} = (Q_{\mathcal{F}}, Q_{\sqcup}, \mathcal{G}, Q_f, \Delta)$ be a NFTAC over $\mathcal{G}$.

- The automaton $\mathcal{A}$ is ***deterministic*** if for each ground flat term t, there is at most one state q such that $t \xrightarrow{*}_{\mathcal{A}} q$.
- The automaton $\mathcal{A}$ is ***complete*** if for each ground flat term t, there a state such that $t \xrightarrow{*}_{\mathcal{A}} q$.
- A state q is ***accessible*** if there is one ground flat term t such that $t \xrightarrow{*}_{\mathcal{A}} q$. The automaton is ***reduced*** if all states are accessible.

4.6.3 Reducing non-determinism

As for usual tree automata, there is an algorithm for computing an equivalent DFTAC from any NFTAC which proves that a language recognized by a NFTAC is also recognized by a DFTAC. The algorithm is similar to the determinization algorithm of the class AWEDC: the ambiguity arising from overlapping constraints is lifted by considering mutually exclusive constraints which cover the original constraints, and using sets of states allows to get rid of the non-determinism of rules having the same left-hand side. Here, we simply have to distinguish between states of $Q_{\mathcal{F}}$ and states of $Q_{\sqcup}$.

```
Determinization algorithm
input $\mathcal{A} = (Q_{\mathcal{F}}, Q_{\sqcup}, \mathcal{G}, Q_f, \Delta)$ a NFTAC.
begin
    A state $[q]$ of the equivalent DFTAC is in $2^{Q_{\mathcal{F}}} \cup 2^{Q_{\sqcup}}$.
    Set $Q^d_{\mathcal{F}} = \emptyset$, $Q^d_{\sqcup} = \emptyset$, $\Delta_d = \emptyset$.
    repeat
      for each $f$ of arity $n$, $[q]_1, \ldots, [q]_n \in Q^d_{\mathcal{F}} \cup Q^d_{\sqcup}$ do
          let $[q] = \{q \mid \exists f(q_1, \ldots, q_n) \to q \in \Delta$ with $q_i \in [q]_i$ for $i = 1, \ldots, n\}$
```

```
        in Set $Q^d_{\mathcal{F}}$ to $Q^d_{\mathcal{F}} \cup \{[q]\}$
           Set $\Delta_d$ to $\Delta_d \cup \{f([q]_1, \dots, [q]_n) \rightarrow [q]\}$
    endfor

    for each $[q] \in Q_{\mathcal{F}}$ do

    for each $[q'] \subseteq \{q'' \mid \exists N.q \xrightarrow{c(N)} q'' \in \Delta$ with $q \in [q]\}$ do
        let $C$ be $\Big(\bigwedge_{q\in[q]} \bigvee_{\substack{N.q \xrightarrow{c_i(N)} q'' \in\Delta \\ q''\in[q']}} c_i(N)\Big) \wedge \Big(\bigwedge_{q\notin[q]} \bigwedge_{\substack{N.q \xrightarrow{c_i(N)} q'' \in\Delta \\ q''\in[q']}} \neg c_i(N)\Big)$
        in Set $Q^d_{\sqcup}$ to $Q^d_{\mathcal{F}} \cup \{[q']\}$
           Set $\Delta_d$ to $\Delta_d \cup \{N.[q] \xrightarrow{C(N)} [q']\}$

    endfor
    endfor

    for each $[q]_1, [q]_2 \in Q^d_{\sqcup}$ do
        let $[q] = \{q \mid \exists q_1 \in [q]_1,\ q_2 \in [q]_2,\ q_1 \sqcup q_2 \rightarrow q_3 \in \Delta\}$
        in Set $Q^d_{\sqcup}$ to $Q^d_{\mathcal{F}} \cup \{[q]\}$
           Set $\Delta_d$ to $\Delta_d \cup \{[q]_1 \sqcup [q]_2 \rightarrow [q]\}$
    endfor

   until no rule can be added to $\Delta_d$
   Set $Q^d_f = \{[q] \in Q^d_{\mathcal{F}} \cup Q^d_{\sqcup} \mid [q] \cap Q_f \neq \emptyset\}$,
end

output: $\mathcal{A}_d = (Q^d_F, Q^d_{\sqcup}, \mathcal{F}, Q^d_f, \Delta_d)$
```

Proposition 29. *The previous algorithm terminates and computes a deterministic flat tree automaton equivalent to the initial one.*

Proof. The termination is obvious. The proof of the correctness relies on the following lemma:

Lemma 6. *$t \xrightarrow{*}_{\mathcal{A}_d} [q]$ if and only if $t \xrightarrow{*}_{\mathcal{A}} q$ for all $q \in [q]$.*

The proof is by structural induction on t and follows the same pattern as the proof for the class AWEDC. □

Therefore we have proved the following theorem stating the equivalence between DFTAC and NFTAC.

Theorem 35. *Let L be a semilinear set of flat terms, then there exists a DFTAC that accepts L.*

4.6.4 Closure Properties of Semilinear Flat Languages

Given an automaton $\mathcal{A} = (Q, \mathcal{G}, Q_f, \Delta)$, it is easy to construct a equivalent complete automaton. If $\mathcal{A}$ is not complete then

- add two new trash states q_t of sort $\mathcal{F}$ and $q_t^{\sqcup}$ of sort $\sqcup$,
- for each $f \in \mathcal{F}$, $q_1, \ldots, q_n \in Q \cup \{q_t, q_t^{\sqcup}\}$, such that there is no rule having $f(q_1, \ldots, q_n)$ as left-hand side, then add $f(q_1, \ldots, q_n) \rightarrow q_t$,
- for each q of sort $\mathcal{F}$, let $c_1(N), \ldots, c_m(N)$ be the conditions of the rules $N.q \xrightarrow{c_i(N)} q'$,
  - if the formula $\exists N\ (c_1(N) \vee \ldots \vee c_m(N))$ is not equivalent to *true*, then add the rule $N.q \xrightarrow{\neg(c_1(N)\vee\ldots\vee c_m(N))(N)} q_t^{\sqcup}$,
  - if there are some q, q' of sort $\sqcup$ such that there is no rule $q \sqcup q' \rightarrow q$", then add the rules $q \sqcup q' \rightarrow q_t^{\sqcup}$ and $q' \sqcup q \rightarrow q_t^{\sqcup}$.
  - if there is some rule $(q_1 \sqcup q_2) \sqcup q_3 \rightarrow q_t^{\sqcup}$ (resp. $q_1 \sqcup (q_2 \sqcup q_3) \rightarrow q_t^{\sqcup}$, add the rule $q_1 \sqcup (q_2 \sqcup q_3) \rightarrow q_t^{\sqcup}$ (resp. $(q_1 \sqcup q_2) \sqcup q_3 \rightarrow q_t^{\sqcup}$) if it is missing.

This last step ensures that we build a flat tree automaton, and it is straightforward to see that this automaton is equivalent to the initial one (same proof as for DFTA). This is stated by the following proposition.

Theorem 36. *For each flat tree automaton $\mathcal{A}$, there exists a complete equivalent automaton $\mathcal{B}$.*

Example 45. The automaton of Example 42 is not complete. It can be completed by adding the states $q_t, q_t^{\sqcup}$, and the rules

$$\begin{array}{lcl} N.q_t & \xrightarrow{N\geq 0} & q_t^{\sqcup} \\ N.q & \xrightarrow{\exists n\ N=2n+1} & q_t^{\sqcup} \\ f(\_,\_) & \longrightarrow & q_t \end{array}$$

where $(\_,\_)$ stands for a pair of $q, q_{\sqcup}, q_t, q_t^{\sqcup}$ such that if a rule the left hand side of which is $f(\_,\_)$ is not already in Δ.

Theorem 37. *The class of semilinear flat languages is closed under union.*

Proof. Let L (resp. M) be a semilinear flat language recognized by $\mathcal{A} = (Q_{\mathcal{F}}, Q_{\sqcup}, \mathcal{G}, Q_f, \Delta)$ (resp. $\mathcal{B} = (Q'_{\mathcal{F}}, Q'_{\sqcup}, \mathcal{G}, Q'_f, \Delta')$), then $L \cup M$ is recognized by $\mathcal{C} = (Q_{\mathcal{F}} \cup Q'_{\mathcal{F}}, Q_{\sqcup} \cup Q'_{\sqcup}, \mathcal{G}, Q_f \cup Q'_f, \Delta \cup \Delta')$. □

Theorem 38. *The class of semilinear flat languages is closed under complementation.*

Proof. Let $\mathcal{A}$ be an automaton recognizing L. Compute a complete automaton $\mathcal{B}$ equivalent to $\mathcal{A}$. Compute a deterministic automaton $\mathcal{C}$ equivalent to $\mathcal{B}$ using the determinization algorithm. The automaton $\mathcal{C}$ is still complete, and we get an automaton recognizing the complement of L by exchanging final and non-final states in $\mathcal{C}$. □

¿From the closure under union and complement, we get the closure under intersection (a direct construction of an automaton recognizing the intersection also exists).

Theorem 39. *The class of semilinear flat languages is closed under intersection.*

4.6.5 Emptiness Decision

The last important property to state is that the emptiness of the language recognized by a flat tree automaton is decidable. The decision procedure relies on an algorithm similar to the decision procedure for tree automata combined to a decision procedure for Presburger's arithmetic. However a straightforward modification of the algorithm in Chapter 1 doesn't work. Assume that the automaton contains the rule $q_1^\sqcup \sqcup q_1^\sqcup \rightarrow q_2^\sqcup$ and assume that there is some flat term t such that $1.t \xrightarrow[\mathcal{A}]{*} q_1^\sqcup$. ¿From these two hypothesis, we can not infer that $1.t \sqcup 1.t \xrightarrow[\mathcal{A}]{*} q_2^\sqcup$ since $1.t \sqcup 1.t$ is not a flat term, contrary to $2.t$. Therefore the decision procedure involves some combinatorics in order to ensure that we always deal with correct flat terms.

¿From now on, let $A = (Q_{\mathcal{F}}, Q_\sqcup, \mathcal{G}, Q_f, \Delta)$ be some given *deterministic* flat tree automaton and let M be the number of states of sort $\sqcup$. First, we need to control the possible infinite number of solutions of Presburger's conditions.

Proposition 30. *There is some computable B such that for each condition $c(N)$ of the rules of $\mathcal{A}$, either each integer n validating c is smaller than B or there are at least $M + 1$ integers smaller than B validating c.*

Proof. First, for each constraint $c(N)$ of a rule of Δ, we check if $c(N)$ has a finite number of solutions by deciding if $\exists P : c(N) \Rightarrow N < P$ is true. If $c(N)$ has a finite number of solutions, it is easy to find a bound $B_1(c(N))$ on these solutions by testing $\exists n : n > k \wedge c(n)$ for $k = 1, 2, \ldots$ until it is false. If $c(N)$ has an infinite number of solutions, one computes the M^{th} solution obtained by checking $\models c(k)$ for $k = 1, 2, \ldots$. We call this M^{th} solution $B_2(c(N))$. The bound B is the maximum of all the $B_1(c(N))$'s and $B_2(c(N))$'s. □

Now we control the maximal width of terms needed to reach a state.

Proposition 31. *For all $t \xrightarrow[\mathcal{A}]{*} q$, there is some $s \xrightarrow[\mathcal{A}]{*} q$ such that for each subterm of s of the form $n_1.v_1 \sqcup \ldots \sqcup n_p.v_p$, we have $p \leq M$ and $n_i \leq B$.*

Proof. The bound on the coefficients n_i is a direct consequence of the previous proposition. The proof on p is by structural induction on t. The only non-trivial case is for $t = m_1.t_1 \sqcup \ldots \sqcup m_k.t_k$. Let us assume that t is the term with the smallest value of k among the terms $\{t' \mid t' \xrightarrow[\mathcal{A}]{*} q\}$.

First we show that $k \leq M$. Let $q_i^\sqcup$ be the states such that $n_i.t_i \rightarrow_{\mathcal{A}} q_i^\sqcup$. We have thus $t \xrightarrow[\mathcal{A}]{*} q_1^\sqcup \sqcup \ldots \sqcup q_k^\sqcup \xrightarrow[\mathcal{A}]{*} q$. By definition of DFTAC, the reduction $q_1^\sqcup \sqcup \ldots \sqcup q_k^\sqcup \xrightarrow[\mathcal{A}]{*} q$ has the form:

$$q_1^\sqcup \sqcup \ldots \sqcup q_k^\sqcup \xrightarrow[\mathcal{A}]{*} q_{[12]}^\sqcup \sqcup q_3^\sqcup \sqcup \ldots \sqcup q_k^\sqcup \xrightarrow[\mathcal{A}]{*} \ldots \xrightarrow[\mathcal{A}]{*} q_{[1\ldots k]}^\sqcup = q$$

for some states $q_{[12]}^\sqcup, \ldots, q_{[1\ldots k]}^\sqcup$ of $Q_\sqcup$.

Assume that $k > M$. The pigeonhole principle yields that $q_{[1,\dots,j_1]} = q_{[1,\dots,j_2]}$ for some $1 \leq j_1 < j_2 \leq k$. Therefore the term

$$t = m_1.t_1 \sqcup \dots \sqcup m_{j_1}.t_{j_1} \sqcup m_{j_2+1}.t_{j_2+1} \sqcup \dots \sqcup m_k.t_k$$

also reaches the state q which contradicts our hypothesis that k is minimal.

Now, it remains only to use the induction hypothesis to replace each t_i by some s_i reaching the same state and satisfying the required conditions. □

A term s such that for all subterm $n_1.v_1 \sqcup \dots n_p.v_p$ of s, we have $p \leq M$ and $n_i \leq B$ will be called *small*. We define some extension $\rightarrow^n_{\mathcal{A}}$ of the move relation by:

- $t \rightarrow^1_{\mathcal{A}} q$ if and only if $t \rightarrow_{\mathcal{A}} q$,
- $t \rightarrow^n_{\mathcal{A}} q$ if and only if $t \xrightarrow{*}_{\mathcal{A}} q$ and
  - either $t = f(t_1,\dots,t_k)$ and for $i = 1,\dots,k$ we have $t_i \rightarrow^{n-1}_{\mathcal{A}} q_i(t_i)$,
  - or $t = n_1.t_1 \sqcup \dots \sqcup n_p.t_p$ and for $i = 1,\dots,p$, we have $t_i \rightarrow^{n-1}_{\mathcal{A}} q_i(t_i)$.

Let $\mathcal{L}^n_q = \{t \rightarrow^p_{\mathcal{A}} q \mid p \leq n \text{ and } t \text{ is small}\}$ with the convention that $\mathcal{L}^0_q = \emptyset$ and $\mathcal{L}_q = \bigcup_{n=1}^{\infty} \mathcal{L}^n_q$. By Proposition 31, $t \rightarrow_{\mathcal{A}} q$ if and only if there is some $s \in \mathcal{L}_q$ such that $s \rightarrow_{\mathcal{A}} q$. The emptiness decision algorithm will compute a finite approximation $\mathcal{R}^n_q$ of these $\mathcal{L}^n_q$ such that $\mathcal{R}^n_q \neq \emptyset$ if and only if $\mathcal{L}^n_q \neq \emptyset$.

Some technical definition is needed first. Let L be a set of flat term, then we define $\parallel L \parallel_P$ as the number of distinct equivalence classes of terms for the $=_P$ relation such that one representant of the class occurs in L. The reader will check easily that the equivalence class of a flat term for the $=_P$ relation is finite.

The decision algorithm is the following one.

```
begin
    for each state $q$ do set $\mathcal{R}^0_q$ to $\emptyset$.
    i=1.
    repeat

     for each state $q$ do  set $\mathcal{R}^i_q$ to $\mathcal{R}^{i-1}_q$
        if $\parallel \mathcal{R}^i_q \parallel_P \leq M$ then
          repeat
               add to $\mathcal{R}^i_q$ all flat terms $t = f(t_1,\dots,t_n)$
               such that $t_j \in \mathcal{R}^{i-1}_{q_j}$, $j \leq n$ and $f(q_1,\dots,q_n) \rightarrow q \in \Delta$
               add to $\mathcal{R}^i_q$ all flat terms $t = n_1.t_1 \sqcup \dots \sqcup n_p.t_p$
               such that $p \leq M$,$n_j \leq B$, $t_j \in \mathcal{R}^{i-1}_{q_j}$ and $n_1.q_1 \sqcup \dots \sqcup n_p.q_p \xrightarrow{*}_{\mathcal{A}} q$.
          until no new term can be added or $\parallel \mathcal{R}^i_q \parallel_P > M$
        endif
        i=i+1
     until $\exists q \in Q_f$ such that $\mathcal{R}^i_q \neq \emptyset$ or $\forall q, \mathcal{R}^i_q = \mathcal{R}^{i-1}_q$

    if $\exists q \in Q_F$ s.t. $\mathcal{R}^i_q \neq \emptyset$
    then return not empty
    else return empty endif
```

end

Proposition 32. *The algorithm terminates after n iterations for some n and $\mathcal{R}_q^n = \emptyset$ if and only if $\mathcal{L}_q = \emptyset$*

Proof. At every iteration, either one $\mathcal{R}_q^i$ increases or else all the $\mathcal{R}_q^i$'s are left untouched in the **repeat** ... **until** loop. Therefore the termination condition will be satisfied after a finite number of iterations, since equivalence classes for $=_P$ are finite.

By construction we have $\mathcal{R}_q^m \subseteq \mathcal{L}_q^m$, but we need the following additional property.

Lemma 7. *For all m,$\mathcal{R}_q^m = \mathcal{L}_q^m$ or $\mathcal{R}_q^m \subseteq \mathcal{L}_q^m$ and $\| \mathcal{R}_q^m \|_P > M$*

The proof is by induction on m.

Base case $m = 0$. Obvious from the definitions.

Induction step. We assume that the property is true for m and we prove that it holds for $m + 1$.
Either $\mathcal{L}_q^m = \emptyset$ therefore $\mathcal{R}_q^m = \emptyset$ and we are done, or $\mathcal{L}_q^m \neq \emptyset$, which we assume from now on.

- $q \in Q_{\mathcal{F}}$.
  - Either there is some rule $f(q_1, \dots, q_n) \rightarrow q$ such that $\mathcal{R}_{q_i}^m \neq \emptyset$ for all $i = 1, \dots, n$ and such that for some q' among $q_1, \dots, q_n$, we have $\| \mathcal{R}_{q'}^m \|_P > M$. Then we can construct at least $M + 1$ terms $t = f(t_1, \dots, t', \dots, t_n)$ where $t' \in \mathcal{R}_{q'}^m$, such that $t \in \mathcal{R}_q^{m+1}$ by giving $M + 1$ non equivalent values to t' (corresponding values for t are also non equivalent). This yields that $\| \mathcal{R}_q^{m+1} \|_P > M$.
  - Or there is no rule as above, therefore $\mathcal{R}_q^{m+1} = \mathcal{L}_q^{m+1}$.
- $q \in Q_{\sqcup}$.

  For each small term $t = n_1.t_1 \sqcup \dots \sqcup n_p.t_p$ such that $t \in \mathcal{L}_q^{m+1}$, there are some terms $s_1, \dots, s_n$ in $\mathcal{R}_{q_i}^m$ such that $t_i \xrightarrow{*}_{\mathcal{A}} q_i$ implies that $s_i \xrightarrow{*}_{\mathcal{A}} q_i$. What we must prove is that $\| \mathcal{R}_{q_i}^m \|_P > M$ for some $i \leq p$ implies $\| \mathcal{R}_q^{m+1} \|_P > M$. Since $\mathcal{A}$ is deterministic, we have that $s \xrightarrow{*}_{\mathcal{A}} q$ and $t \xrightarrow{*}_{\mathcal{A}} q'$ with $q \neq q'$ implies that $s \neq_P t$. Let S be the set of states occurring in the sequence $q_1, \dots, q_p$. We prove by induction on the cardinal of S that if there is some q_i such that $\| \mathcal{R}_{q_i}^m \|_P > M$, we can build at least $M + 1$ terms in $\mathcal{R}_q^{m+1}$ otherwise we build at least one term of $\mathcal{R}_q^{m+1}$.

  Base case $S = \{q'\}$, and therefore all the q_i are equal to q'. Either $\| \mathcal{R}_{q'}^m \|_P \leq M$ and we are done or $\| \mathcal{R}_{q'}^m \|_P > M$ and we know that there are $s_1, \dots, s_{M+1}, \dots$ pairwise non equivalent terms reaching q'. Therefore, there are at least $\binom{M+1}{M} \geq M + 1$ different non equivalent possible terms $n_{i_1}.s_{i_1} \sqcup \dots \sqcup n_{i_p}.s_{i_p}$. Moreover each of these terms S satisfies $s \xrightarrow{m+1}_{\mathcal{A}} q$, which proves the result.

Induction step. Let $S = S' \cup \{q'\}$ where the property is true for S'. We can assume that $\| \mathcal{R}^m_{q'} \|_P \leq M$ (otherwise all $\| \mathcal{R}^m_{q_i} \|_P$ are less than or equal to M).

Let $i_1, \dots, i_k$ be the positions of q' in $q_1, \dots, q_p$, let $j_1, \dots, j_l$ be the positions of the states different from q' in $q_1, \dots, q_p$. By induction hypothesis, there are some flat terms s_j such that $n_{j_1}.s_{j_1} \sqcup \dots \sqcup n_{j_l}.s_{j_l}$ is a valid flat term. Since $\mathcal{A}$ is deterministic and q' is different from all element of S', we know that $s_i \neq_P s_j$ for any $i \in \{i_1, \dots, i_k\}$, $j \in \{j_1, \dots, j_k\}$. Therefore, we use the same reasoning as in the previous case to build at least $C^k_{M+1} \geq M+1$ pairwise non equivalent flat terms $s = n_1.s_1 \sqcup \dots \sqcup n_p.s_p$ such that $s \rightarrow^{m+1}_{\mathcal{A}} q$.

The termination of the algorithm implies that for each $m \geq n$, $\mathcal{R}^m_q = \mathcal{L}^m_q$ or $\mathcal{R}^m_q \subseteq \mathcal{L}^m_q$ and $\| \mathcal{R}^m_q \|_P > M$. Therefore $\mathcal{L}_q = \emptyset$ if and only if $\mathcal{R}^n_q = \emptyset$.

□

The following theorem summarizes the previous results.

Theorem 40. *Let $\mathcal{A}$ be a DFTAC, then it is decidable whether the language accepted by $\mathcal{A}$ is empty or not.*

The reader should see that the property that $\mathcal{A}$ *deterministic* is crucial in proving the emptiness decision property. Therefore proving the emptiness of the language recognized by a NFTAC implies to compute an equivalent *DFTAC* first.

Another point is that the previous algorithm can be easily modified to compute the set of accessible states of $\mathcal{A}$.

4.7 Exercises

Exercise 53.

1. Show that the automaton $\mathcal{A}_+$ of Example 35 accepts only terms of the form $f(t_1, s^n(0), s^m(0), s^{n+m}(0))$
2. Conversely, show that, for every pair of natural numbers (n, m), there exists a term t_1 such that $f(t_1, s^n(0), s^m(0), s^{n+m}(0))$ is accepted by $\mathcal{A}_+$.
3. Construct an automaton $\mathcal{A}_\times$ of the class AWEDC which has the same properties as above, replacing $+$ with $\times$
4. Give a proof that emptiness is undecidable for the class AWEDC, reducing Hilbert's tenth problem.

Exercise 54. Give an automaton of the class AWCBB which accepts the set of terms t (over the alphabet $\{a(0), b(0), f(2)\}$) having a subterm of the form $f(u, u)$. (i.e the set of terms that are reducible by a rule $f(x, x) \rightarrow v$).

Exercise 55. Show that the class AWCBB is not closed under linear tree homomorphisms. Is it closed under inverse image of such morphisms ?

Exercise 56. Give an example of two automata in AWCBB such that the set of pairs of terms recognized respectively by the automata is not itself a member of AWCBB.

Exercise 57. (Proposition 24) Show that the class of (languages recognized by) reduction automata is closed under intersection and union. Show that the set of balanced term on alphabet $\{a, f\}$ is not recognizable by a reduction automaton, showing that the class of languages recognized by) reduction automata is not closed under complement.

Exercise 58. Show that the class of languages recognized by reduction automata is preserved under linear tree homomorphisms. Show however that this is no longer true for arbitrary tree homomorphisms.

Exercise 59. Let $\mathcal{A}$ be a reduction automaton. We define a ternary relation $q \xrightarrow{w} q'$ contained in $Q \times \mathbb{N}^* \times Q$ as follows:

- for $i \in \mathbb{N}$, $q \xrightarrow{i} q'$ if and only if there is a rule $f(q_1, \dots, q_n) \xrightarrow[\mathcal{A}]{c} q'$ with $q_i = q$
- $q \xrightarrow{i \cdot w} q'$ if and only if there is a state q'' such that $q \xrightarrow{i} q''$ and $q'' \xrightarrow{w} q'$.

Moreover, we say that a state $q \in Q$ is a *constrained state* if there is a rule $f(q_1, \dots, q_n) \xrightarrow[\mathcal{A}]{c} q$ in $\mathcal{A}$ such that c is not a valid constraint.

We say that the *the constraints of* $\mathcal{A}$ *cannot overlap* if, for each rule $f(q_1, \dots, q_n) \xrightarrow{c} q$ and for each equality (resp. disequality) $\pi = \pi'$ of c, there is no strict prefix p of π and no constrained state q' such that $q' \xrightarrow{p} q$.

1. Consider the rewrite system on the alphabet $\{f(2), g(1), a(0)\}$ whose left members are $f(x, g(x)), g(g(x)), f(a, a)$. Compute a reduction automaton, whose constraints do not overlap and which accepts the set of irreducible ground terms.
2. Show that emptiness can be decided in polynomial time for reduction automata whose constraints do not overlap. (Hint: it is similar to the proof of Theorem 30.)
3. Show that any language recognized by a reduction automaton whose constraints do not overlap is an homomorphic image of a language in the class AWCBB. Give an example showing that the converse is false.

Exercise 60. Prove the Proposition **??** along the lines of Proposition 15.

Exercise 61. The purpose of this exercise is to give a construction of an automaton with disequality constraints (no equality constraints) whose emptiness is equivalent to the ground reducibility of a given term t with respect to a given term rewriting system $\mathcal{R}$.

1. Give a direct construction of an automaton with disequality constraints $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ which accepts the set of irreducible ground terms
2. Show that the class of languages recognized by automata with disequality constraints is closed under intersection. Hence the set of irreducible ground instances of a linear term is recognized by an automaton with disequality constraints.
3. Let $\mathcal{A}_{\mathrm{NF}(\mathcal{R})} = (Q_{\mathrm{NF}}, \mathcal{F}, Q^f_{\mathrm{NF}}, \Delta_{\mathrm{NF}})$. We compute $\mathcal{A}_{\mathrm{NF},t} \stackrel{\mathrm{def}}{=} (Q_{\mathrm{NF},t}, \mathcal{F}, Q^f_{\mathrm{NF},t}, \Delta_{\mathrm{NF},t})$ as follows:
   - $Q_{\mathrm{NF},t} \stackrel{\mathrm{def}}{=} \{t\sigma|_p \mid p \in \mathit{Pos}(t)\} \times Q_{\mathrm{NF}}$ where σ ranges over substitutions from $\mathit{NLV}(t)$ (the set of variables occurring at least twice in t) into Q^f_{NF}.
   - For all $f(q_1, \dots, q_n) \xrightarrow{c} q \in \Delta_{\mathrm{NF}}$, and all $u_1, \dots, u_n \in \{t\sigma|_p \mid p \in \mathit{Pos}(t)\}$, $\Delta_{\mathrm{NF},t}$ contains the following rules:

- $f([q_{u_1}, q_1], \ldots, [q_{u_n}, q_n]) \xrightarrow{c \wedge c'} [q_{f(u_1,\ldots,u_n)}, q]$ if $f(u_1, \ldots, u_n) = t\sigma_0$ and c' is constructed as sketched below.
- $f([q_{u_1}, q_1], \ldots, [q_{u_n}, q_n]) \xrightarrow{c} [q_{f(u_1,\ldots,u_n)}, q]$ if $[q_{f(u_1,\ldots,u_n)}, q] \in Q_{\mathrm{NF},t}$ and we are not in the first case.
- $f([q_{u_1}, q_1], \ldots, [q_{u_n}, q_n]) \xrightarrow{c} [q_q, q]$ in all other cases

c' is constructed as follows. From $f(u_1, \ldots, u_n)$ we can retrieve the rules applied at position p in t. Assume that the rule at p checks $\pi_1 \neq \pi_2$. This amounts to check $p\pi_1 \neq p\pi_2$ at the root position of t. Let $\mathcal{D}$ be all disequalities $p\pi_1 \neq p\pi_2$ obtained in this way. The non linearity of t implies some equalities: let $\mathcal{E}$ be the set of equalities $p_1 = p_2$, for all positions p_1, p_2 such that $t|_{p_1} = t|_{p_2}$ is a variable. Now, c' is the set of disequalities $\pi \neq \pi'$ which are not in $\mathcal{D}$ and that can be inferred from $\mathcal{D}, \mathcal{E}$ using the rules

$$\begin{array}{l} pp_1 \neq p_2,\ p = p' \ \vdash\ p'p_1 \neq p_2 \\ p \neq p',\ pp_1 = p_2 \ \vdash\ p'p_1 \neq p_2 \end{array}$$

For instance, let $t = f(x, f(x, y))$ and assume that the automaton $\mathcal{A}_{\mathrm{NF}}$ contains a rule $f(q, q) \xrightarrow{1 \neq 2} q$. Then the automaton $\mathcal{A}_{\mathrm{NF},t}$ will contain the rule $f([q_q, q], [q_{f(q,q)}, q]) \xrightarrow{1 \neq 2 \wedge 1 \neq 22} q$.

The final states are $[q_u, q_f]$ where $q_f \in Q^f_{\mathrm{NF}}$ and u is an instance of t.
Prove that $\mathcal{A}_{\mathrm{NF},t}$ accepts at least one term if and only if t is not ground reducible by $\mathcal{R}$.

Exercise 62. Prove Theorem 34 along the lines of the proof of Theorem 24.

Exercise 63. Show that the algorithm for deciding emptiness of deterministic complete flat tree automaton works for non-deterministic flat tree automata such that for each state q the number of non-equivalent terms reaching q is 0 or greater than or equal to 2.

Exercise 64. (Feature tree automata)
Let $\mathcal{F}$ be a finite set of feature symbols (or attributes) denoted by $f, g, \ldots$ and $\mathcal{S}$ be a set of constructor symbols (or sorts) denoted by $A, B, \ldots$. In this exercise and the next one, a tree is a rooted directed acyclic graph, a multitree is a tree such that the nodes are labeled over $\mathcal{S}$ and the edges over $\mathcal{F}$. A multitree is either $(A, \emptyset)$ or (A, E) where E is a finite multiset of pairs (f, t) with f a feature and t a multitree. A feature tree is a multitree such that the edges outgoing from the same node are labeled by different features. The $+$ operation takes a multitree $t = (A, E)$, a feature f and a multitree t' to build the multitree $(A, E \cup (f, t'))$ denoted by $t + ft'$.

1. Show that $t + f_1t_1 + f_2t_2 = t + f_2t_2 + f_1t_1$ (OI axiom: order independence axiom) and that the algebra of multitrees is isomorphic to the quotient of the free term algebra over $\{+\} \cup \mathcal{F} \cup \mathcal{S}$ by OI.
2. A deterministic $\mathcal{M}$-automaton is a triple $(\mathcal{A}, h, Q_f)$ where $\mathcal{A}$ is an finite $\{+\} \cup \mathcal{F} \cup \mathcal{S}$-algebra, $h : \mathcal{M} \to \mathcal{A}$ is a homomorphism, Q_f (the final states) is a subset of the set of the values of sort $\mathcal{M}$. A tree is accepted if and only if $h(t) \in Q_f$.
   (a) Show that a $\mathcal{M}$-automaton can be identified with a bottom-up tree automaton such that all trees equivalent under OI reach the same states.
   (b) A feature tree automaton is a $\mathcal{M}$-automaton such that for each sort s ($\mathcal{M}$ or $\mathcal{F}$), for each q the set of the c's of arity 0 interpreted as q in $\mathcal{A}$ is finite or co-finite. Give a feature tree to recognize the set of natural numbers where n is encoded as $(0, \{suc, (0, \{\ldots, (0, \emptyset)\})\})$ with n edges labeled by suc.

(c) Show that the class of languages accepted by feature tree automata is closed under boolean operations and that the emptiness of a language accepted by a feature automaton is decidable.

(d) A non-deterministic feature tree automaton is a tuple (Q, P, h, Q_f) such that Q is the set of states of sort $\mathcal{M}$, P the set of states of sort $\mathcal{F}$, h is composed of three functions $h_1 : \mathcal{S} \to 2^Q$, $h_2 : \mathcal{F} \to 2^P$ and the transition function $+ : Q \times P \times Q \to 2^Q$. Moreover $q + p_1q_1 + p_2q_2 = q + p_2q_2 + p_1q_1$ for each q, q_1, q_2, p_1, p_2, $\{s \in \mathcal{S} \mid p \in h_1(s)\}$ and $\{f \in \mathcal{F} \mid p \in h_2(f)\}$ are finite or co-finite for each p. Show that any non-deterministic feature tree automaton is equivalent to a deterministic feature tree automaton.

Exercise 65. (Characterization of recognizable flat feature languages)
A flat feature tree is a feature tree of depth 1 where depth is defined by $depth((A, \emptyset)) = 0$ and $depth((A, E)) = 1 + max\{depth(t) \mid (f, t) \in E\}$. Counting constraints are defined by:

$$\begin{aligned} C(x) \quad ::= \quad & card(\varphi \in F \mid \exists y.(x\varphi y) \wedge Ty\}) = n \ mod \ m \\ & \mid Sx \\ & \mid C(x) \vee C(x) \\ & \mid C(x) \wedge C(x) \end{aligned}$$

where n, m are integers, S and T finite or co-finite subsets of $\mathcal{S}$, F a finite or co-finite subset of $\mathcal{F}$ and $n \ mod \ 0$ is defined as n. The semantics of the first type of constraint is: $C(x)$ holds if the number of edges of x going from the root to a node labeled by a symbol of T is equal to $n \ mod \ m$. The semantics of Sx is: Sx holds if the root of x is labeled by a symbol of S.

1. Show that the constraints are closed under negation. Show that the following constraints can be expressed in the constraint language (F is a finite subset of $\mathcal{F}$, $f \in F$, $A \in \mathcal{S}$): there is one edge labeled f from the root, a given finite subset of $\mathcal{F}$. There is no edge labeled f from the root, the root is labeled by A.
2. A set L of flat multitrees is counting definable if and only if there some counting constraint C such that $L = \{x \mid C(x) \ holds\}$. Show that a set of flat feature trees is counting definable if and only if it is recognizable by a feature tree automaton. hint: identify flat trees with multisets over $(\mathcal{F} \cup \{root\}) \times \mathcal{S}$ and $+$ with multiset union.

4.8 Bibliographic notes

RATEG appeared in Mongy's thesis [Mon81]. Unfortunately, as shown in [Mon81] the emptiness problem is undecidable for the class RATEG (and hence for AWEDC). The undecidability can be even shown for a more restricted class of *automata with equality tests between cousins* (see [Tom92]).
The remarkable subclass AWCBB is defined in [BT92]. This paper presents the results cited in section 4.3, especially theorem 30.
Concerning complexity, the result used in section 4.3.2 (EXPTIME-completeness of the emptiness of the intersection of n recognizable tree languages) may be found in [FSVY91, Sei94b].
[DCC95] is concerned with reduction automata and their use as a tool for the decision of the encompassment theory in the general case.
The first decidability proof for ground reducibility is due to [Pla85]. In [?], ground reducibility decision is shown EXPTIME-complete. In this work, an EXPTIME algorithm for emptiness decision for AWEDC with only disequality constrained The result mentioned in section 4.5.

The class of generalized reduction automata is introduced in [CCC$^+$94]. In this paper, a efficient cleaning algorithm is given for emptiness decision.

There have been many work dealing with automata where the width of terms is not bounded. In [Cou89], Courcelle devises an algebraic notion of recognizability and studies the case of equational theories. Then he gives several equational theories corresponding to several notions of trees like ordered or unordered, ranked or unranked trees and provides the tree automata to accept these objects. Actually the axioms used for defining these notions are commutativity (for unordered) or associativity (for unranked) and what is needed is to build tree automata such that all element of the same equivalence class reach the same state. Trees can be also defined as finite, acyclic rooted ordered graphs of bounded degree. Courcelle [Cou92] has devised a notion of of recognizable set of graphs and suggests to devise graph automata for accepting recognizable graphs of bounded tree width. He gives such automata for trees defined as unbounded, unordered, undirected, unrooted trees (therefore these are not what we call tree in this book). Actually, he shows that recognizable sets of graphs are (homomorphic image of) sets of equivalence class of terms, where the equivalence relation is the congruence induced by a set of equational axioms including associativity-commutativity axiom and identity element. He gives several equivalent notions for recognizability from which he gets the definitions of automata for accepting recognizable languages.

Feature tree are a generalization of first-order trees introduced for modeling record structures. A feature tree is a finite tree whose nodes are labelled by constructor symbols and edges are labelled by feature symbols Niehren and Podelski [NP93] have studied the algebraic structures of feature trees and have devised feature tree automata for recognizing sets of feature trees. They have shown that this class of feature trees enjoy the same properties as regular tree language and give a characterization of these sets using counting constraints which are a subset of Presburger's arithmetic formulas. See exercise 64 for more details.

Chapter 5

Tree Set Automata

5.1 Introduction

"The notion of type expresses the fact that one just cannot apply any operator to any value. Inferring and checking a program's type is then a proof of partial correction" quoting Marie-Claude Gaudel. *"The main problem in this field is to be flexible while remaining rigorous, that is to allow polymorphism (a value can have more than one type) in order to avoid repetitions and write very general programs while preserving decidability of their correction with respect to types."*

On that score, the set constraints formalism is a compromise between power of expression and decidability. This has been the object of active research for a few years.

Set constraints are relations between sets of terms. For instance, let us define the natural numbers with 0 and the successor relation denoted by s. Thus, the constraint

$$\mathsf{Nat} = 0 \cup s(\mathsf{Nat}) \tag{5.1}$$

corresponds to this definition. Let us consider the following system:

$$\begin{array}{rcl} \mathsf{Nat} & = & 0 \cup s(\mathsf{Nat}) \\ \mathsf{List} & = & cons(\mathsf{Nat}, \mathsf{List}) \cup \mathsf{nil} \\ \mathsf{List}_+ & \subseteq & \mathsf{List} \\ \mathsf{car}(\mathsf{List}_+) & \subseteq & s(\mathsf{Nat}) \end{array} \tag{5.2}$$

The first constraint defines natural numbers. The second constraint codes the set of LISP-like lists of natural numbers. The empty list is nil and other lists are obtained the constructor symbol cons. The last two constraints represent the set of lists with a non zero first element.

Set constraints are the essence of Set Based Analysis. The basic idea is to reason about program variables as sets of possible values. Set Based Analysis involves first writing set constraints expressing relationships between sets of program values, and then solving the system of set constraints. A single approximation is: all dependencies between the values of program variables are ignored. Techniques developed for Set Based Analysis have been successfully applied in program analysis and type inference and the technique can be combined with others [HJ92].

Set constraints have also been used to define a constraint logic programming language over sets of ground terms that generalizes ordinary logic programming over an Herbrand domain [Koz94].

In a more general way, a *system of set constraints* is a conjunction of *positive* constraints of the form $exp \subseteq exp'$[1] and *negative* constraints of the form $exp \not\subseteq exp'$. Right hand side and left hand side of these inequalities are *set expressions*, which are built with

- function symbols: in our example 0, s, cons, nil are function symbols.
- operators: union $\cup$, intersection $\cap$, complement $\sim$
- projection symbols: for instance, in the last equation of system (5.2) car denotes the first component of cons. In the set constraints syntax, this is written $\mathsf{cons}^{-1}_{(1)}$.
- set variables like Nat or List.

An interpretation assigns to each set variable a set of terms only built with function symbols. A solution is an interpretation which satisfies the system. For example, $\{0, s(0), s(s(0)), \ldots\}$ is a solution of Equation (5.1).

In the set constraint formalism, set inclusion and set union express in a natural way parametric polymorphism: $\mathsf{List} \subseteq \mathsf{nil} \cup \mathsf{cons}(X, \mathsf{List})$.

In logic or functional programming, one often use dynamic procedures to deal with type. In other words, a run-time procedure checks whether or not an expression is well-typed. This permits maximum programming flexibility at the potential cost of efficiency and security. Static analysis partially avoids these drawbacks with the help of type inference and type checking procedures. The information extracted at compile time is also used for optimization.

Basically, program sources are analyzed at compile time and an ad hoc formalism is used to represent the result of the analysis. For types considered as sets of values, the set constraints formalism is well suited to represent them and to express their relations. Numerous inference and type checking algorithms in logic, functional and imperative programming are based on a resolution procedure for set constraints.

Most of the earliest algorithms consider systems of set constraints with weak power of expression. More often than not, these set constraints always have a least solution — w.r.t. inclusion — which corresponds to a (tuple of) regular set of terms. In this case, types are usual sorts. A sort signature defines a tree automaton (see Section 3.4.1 for the correspondence between automata and sorts). Therefore, these methods are closely related finite tree automata and use classical algorithms on these recognizers, like the ones presented in Chapter 1.

In order to obtain a more precise information, it is necessary to enrich the set constraints vocabulary. In one hand, with a large vocabulary an analysis

[1] $exp = exp'$ for $exp \subseteq exp' \wedge exp' \subseteq exp$.

can be accurate and relevant, but on the other hand, solutions are difficult to obtain.

Nonetheless, an essential property must be preserved: the decidability of satisfiability. There must exists a procedure which determines whether or not a system of set constraints has solutions. In other words, extracted information must be sufficient to say whether the objects of an analyzed program have a type. It is crucial, therefore, to know which classes of set constraints are decidable, and identifying the complexity of set constraints is of paramount importance.

A second important characteristic to preserve is to represent solutions in a convenient way. We want to obtain a kind of solved form from which one can decide whether a system has solutions and one can "compute" them.

In this chapter, we present an automata-based algorithm for solving systems of positive and negative set constraints where no projection symbols occurs. We define a new class of automata recognizing sets of (codes of) n-tuples of tree languages. Given a system of set constraints, there exists an automaton of this class which recognizes the set of solutions of the system. Therefore properties of our class of automata directly translate to set constraints.

In order to introduce our automata, we discuss the case of unary symbols, i.e. the case of strings over finite alphabet. For instance, let us consider the following constraints over the alphabet composed of two unary symbols a and b and a constant 0:

$$\begin{aligned} Xaa \cup Xbb &\subseteq X \\ Y &\subseteq X \end{aligned} \tag{5.3}$$

This system of set constraints can be encoded in a formula of the monadic second order theory of 2 successors named a and b:

$$\begin{aligned} &\forall u\ (u \in X \Rightarrow (uaa \in X \wedge ubb \in X)) \wedge \\ &\forall u\ u \in X \Rightarrow u \in Y \end{aligned}$$

We have depicted in Fig 5.1 (a beginning of) an infinite tree which is a model of the formula. Each node of the tree is labelled with a couple of points. The two components correspond to sets X and Y. A black point in the first component means that the current node belongs to X. Conversely, a white point in the first component means that the current node does not belong to X. Here we have $X = \{\varepsilon, aa, bb, \dots\}$ and $Y = \{\varepsilon, bb, \dots\}$.

Such a tree language is Rabin-recognizable by a tree automaton which must avoid forbidden patterns depicted in Figure 5.2.

Given a ranked alphabet of unary symbols and one constant and a system of set constraints over $\{X_1, \dots, X_n\}$, one can encode a solution with a $\{0,1\}^n$-valued infinite tree and the set of solutions is recognized by an infinite tree automaton. Therefore, decidability of satisfiability of systems of set constraints can easily be derived from Rabin's Tree Theorem [Rab69] because infinite tree automata can be considered as an acceptor model for n-tuples of word languages over finite alphabet[2].

[2]The entire class of Rabin's tree languages is not captured by solutions of set of words constraints. Set of words constraints define a class of languages which is strictly smaller than Büchi tree automata.

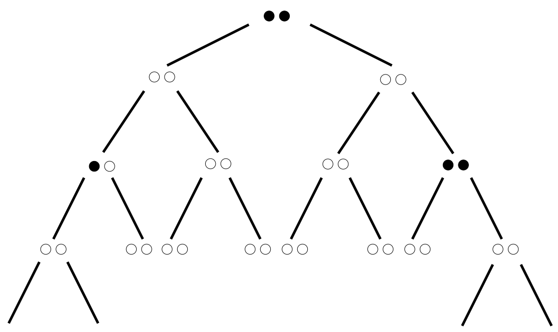


Figure 5.1: A solution of the set constraint 5.3

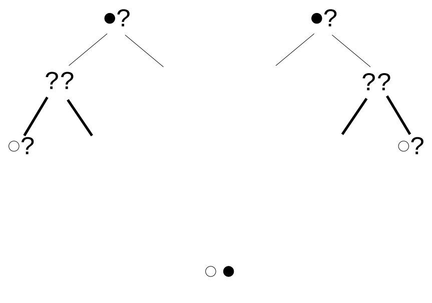


Figure 5.2: The set of forbidden patterns

We extend this method to set constraints with symbols of arbitrary arity. Therefore, we define an acceptor model for mappings from $T(\mathcal{F})$, where $\mathcal{F}$ is a ranked alphabet, into a set $E = \{0,1\}^n$ of labels. Our automata can be viewed as an extension of infinite tree automata, but we will use weaker acceptance condition. The acceptance condition is: the range of a successful run is in a specified set of accepting set of states. We will prove that we can design an automaton which recognizes the set of solutions of a system of both positive and negative set constraints. For instance, let us consider the following system:

$$Y \not\subseteq \perp \tag{5.4}$$
$$X \subseteq f(Y, \sim X) \cup a \tag{5.5}$$

where $\perp$ stands for the empty set and $\sim$ stands for the complement symbol.

The underlying structure is different than in the previous example since it is now the whole set of terms on the alphabet composed of a binary symbol f and a constant a. Having a representation of this structure in mind is non trivial. One can imagine a directed graph whose vertices are terms and such that there exists an edge between each couple of terms in the direct subterm relation (see figure 5.3).

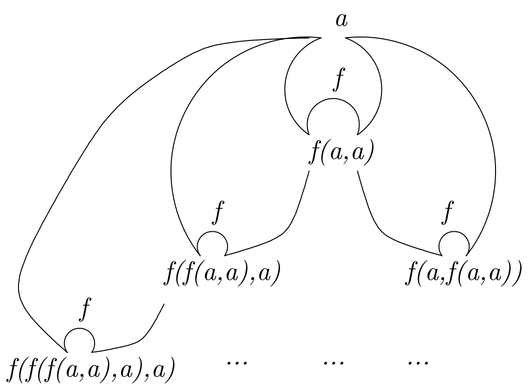


Figure 5.3: The (beginning of the) underlying structure for a two letter alphabet $\{f(,), a\}$.

An automaton have to associate a state with each node following a finite set of rules. In the case of the example above, states are also couples of $\bullet$ or $\circ$.

Each vertex is of infinite out-degree, nonetheless one can define as in the word case forbidden patterns for incoming vertices which such an automaton have to avoid in order to satisfy Eq. (5.5) (see Fig. 5.4, Pattern ? stands for $\bullet$ or $\circ$). The acceptance condition is illustrated using Eq. (5.4). Indeed, to describe a solution of the system of set constraints, the pattern ?$\bullet$ must occur somewhere in a successful “run” of the automaton.

Consequently, decidability of systems of set constraints is a consequence of decidability of emptiness in our class of automata. Emptiness decidability is easy for automata without acceptance conditions (it corresponds to the case of positive set constraints only). The proof is more difficult and technical in the general case and is not presented here. Moreover, and this is the main advantage

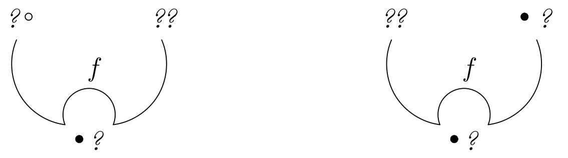


Figure 5.4: Forbidden patterns for (5.5).

of an automaton-based method, properties of recognizable sets directly translate to sets of solutions of systems of set constraints. Therefore, we are able to prove nice properties. For instance, we can prove that a non empty set of solutions always contain a regular solution. Moreover we can prove the decidability of existence of finite solutions.

5.2 Definitions and examples

Infinite tree automata are an acceptor model for infinite trees, i.e. for mappings from A^* into E where A is a finite alphabet and E is a finite set of labels. We define and study $\mathcal{F}$-generalized tree set automata which are an acceptor model for maps from $T(\mathcal{F})$ into E where $\mathcal{F}$ is a finite ranked alphabet and E is a finite set of labels.

5.2.1 Generalized tree sets

Let $\mathcal{F}$ be a ranked alphabet and E be a finite set. An ***E-valued $\mathcal{F}$-generalized tree set*** g is a map from $T(\mathcal{F})$ into E. We denote by $\mathcal{G}_E$ the set of E-valued $\mathcal{F}$-generalized tree sets.

For the sake of brevity, we do not mention the signature $\mathcal{F}$ which strictly speaking is in order in generalized tree sets. We also use the abbreviation GTS for generalized tree sets.

Throughout the chapter, if $c \in \{0,1\}^n$, then c_i denotes the i^{th} component of the tuple c.

If we consider the set $E = \{0,1\}^n$ for some n, a generalized tree set g in $\mathcal{G}_{\{0,1\}^n}$ can be considered as a n-tuple $(L_1, \ldots, L_n)$ of tree languages over the ranked alphabet $\mathcal{F}$ where $L_i = \{t \in T(\mathcal{F}) \mid g(t)_i = 1\}$.

We will need in the chapter the following operations on generalized tree sets. Let g (resp. g') be a generalized tree set in $\mathcal{G}_E$ (resp. $\mathcal{G}_{E'}$). The generalized tree set $g \uparrow g' \in \mathcal{G}_{E\times E'}$ is defined by $g \uparrow g'(t) = (g(t), g'(t))$, for each term t in $T(\mathcal{F})$. Conversely let g be a generalized tree set in $\mathcal{G}_{E\times E'}$ and consider the projection π from $E \times E'$ into the E-component then $\pi(g)$ is the generalized tree set in $\mathcal{G}_E$ defined by $\pi(g)(t) = \pi(g(t))$. Let $G \subseteq \mathcal{G}_{E\times E'}$ and $G' \subseteq \mathcal{G}_E$, then $\pi(G) = \{\pi(g) \mid g \in G\}$ and $\pi^{-1}(G') = \{g \in \mathcal{G}_{E\times E'} \mid \pi(g) \in G'\}$.

5.2.2 Tree Set Automata

A ***generalized tree set automaton*** $\mathcal{A} = (Q, \Delta, \Omega)$ (GTSA) over a finite set E consist of a finite state set Q, a transition relation $\Delta \subseteq \bigcup_p Q^p \times \mathcal{F}_p \times E \times Q$ and a set $\Omega \subseteq 2^Q$ of accepting sets of states.

A ***run*** of $\mathcal{A}$ (or $\mathcal{A}$-run) on a generalized tree set $g \in \mathcal{G}_E$ is a map $r : T(\mathcal{F}) \to Q$ with :

$$(r(t_1), \ldots , r(t_p), f, g(f(t_1, \ldots , t_p)), r(f(t_1, \ldots , t_p))) \in \Delta$$

for $t_1, \ldots , t_p \in T(\mathcal{F})$ and $f \in \mathcal{F}_p$. The run r is ***successful*** if the range of r is in Ω i.e. $r(T(\mathcal{F})) \in \Omega$.

A generalized tree set $g \in \mathcal{G}_E$ is ***accepted*** by the automaton $\mathcal{A}$ if some run r of $\mathcal{A}$ on g is successful. We denote by $\mathcal{L}(\mathcal{A})$ the set of E-valued generalized tree sets accepted by a generalized tree set automaton $\mathcal{A}$ over E. A set $G \subseteq \mathcal{G}_E$ is recognizable if $G = \mathcal{L}(\mathcal{A})$ for some generalized tree set automaton $\mathcal{A}$.

In the following, a rule $(q_1, \ldots, q_p, f, l, q)$ is also denoted by $f(q_1, \ldots, q_p) \xrightarrow{l} q$. Consider a term $t = f(t_1, \ldots, t_p)$ and a rule $f(q_1, \ldots, q_p) \xrightarrow{l} q$, this rule can be applied in a run r on a generalized tree set g for the term t if $r(t_1) = q_1$, $\ldots$,$r(t_p) = q_p$, t is labeled by l, i.e. $g(t) = l$. If the rule is applied, then $r(t) = q$.

A generalized tree set automaton $\mathcal{A} = (Q, \Delta, \Omega)$ over E is

- ***deterministic*** if for each tuple $(q_1, \ldots , q_p, f, l) \in Q^p \times \mathcal{F}_p \times E$ there is at most one state $q \in Q$ such that $(q_1, \ldots, q_p, f, l, q) \in \Delta$.
- ***strongly deterministic*** if for each tuple $(q_1, \ldots, q_p, f) \in Q^p \times \mathcal{F}_p$ there is at most one pair $(l, q) \in E \times Q$ such that $(q_1, \ldots, q_p, f, l, q) \in \Delta$.
- ***complete*** if for each tuple $(q_1, \ldots , q_p, f, l) \in Q^p \times \mathcal{F}_p \times E$ there is at least one state $q \in Q$ such that $(q_1, \ldots, q_p, f, l, q) \in \Delta$.
- ***simple*** if Ω is "subset-closed", that is $\omega \in \Omega \Rightarrow (\forall \omega' \subseteq \omega \ \omega' \in \Omega)$.

Successfulness for simple automata just implies some states are *not* assumed along a run. For instance, if the accepting set of a GTSA $\mathcal{A}$ is $\Omega = 2^Q$ then $\mathcal{A}$ is simple and any run is successful. But, if $\Omega = \{Q\}$, then $\mathcal{A}$ is not simple and each state must be assumed at least once in a successful run. The definition of simple automata will be clearer with the relationships with set constraints and the emptiness property (see Section 5.4). Briefly, positive set constraints are connected with simple GTSA for which the proof of emptiness decision is straightforward. Another and equivalent definition for simple GTSA relies on the acceptance condition : a run r is successful if and only if $r(T(\mathcal{F})) \subseteq \omega \in \Omega$.

There is in general an *infinite* number of runs — and hence an *infinite* number of GTS recognized — even in the case of deterministic generalized tree set automata (see example 46.2). Nonetheless, given a GTS g, there is at most one run on g for a deterministic generalized tree set automata. But, in the case of *strongly* deterministic generalized tree set automata, there is at most one run (see example 46.1) and therefore there is at most one GTS recognized.

Example 46.

Ex. 46.1 Let $E = \{0, 1\}$, $\mathcal{F} = \{\mathsf{cons}(,), s(), \mathsf{nil}, 0\}$. Let $\mathcal{A} = (Q, \Delta, \Omega)$ be defined by $Q = \{\mathsf{Nat}, \mathsf{List}, \mathsf{Term}\}$, $\Omega = 2^Q$, and Δ is the following set of rules:

$$0 \xrightarrow{0} \mathsf{Nat} \ ; \ s(\mathsf{Nat}) \xrightarrow{0} \mathsf{Nat} \ ; \quad nil \xrightarrow{1} \mathsf{List} \quad ;$$
$$\mathsf{cons}(\mathsf{Nat}, \mathsf{List}) \xrightarrow{1} \mathsf{List} \ ;$$
$$\mathsf{cons}(q, q') \xrightarrow{0} \mathsf{Term} \quad \forall (q, q') \neq (\mathsf{Nat}, \mathsf{List}) \ ;$$
$$s(q) \xrightarrow{0} \mathsf{Term} \quad \forall q \neq \mathsf{Nat} \ .$$

$\mathcal{A}$ is strongly deterministic, simple, and not complete. $\mathcal{L}(\mathcal{A})$ is a singleton set reduced to a generalized tree set $g \in \mathcal{G}_{\{0,1\}^n}$. Indeed, there is a unique run r on a unique generalized tree set g. The run r maps every natural number on state Nat, every list on state List and the other terms on state Term. Therefore g maps a natural number on 0, a list on 1 and the other terms on 0. Hence, we say that $\mathcal{L}(\mathcal{A})$ is the regular tree language L of Lisp-like lists of natural numbers.

Ex. 46.2 Let $E = \{0, 1\}$, $\mathcal{F} = \{\mathsf{cons}(,), s(), \mathsf{nil}, 0\}$, and let $\mathcal{A}' = (Q', \Delta', \Omega')$ be defined by $Q' = Q$, $\Omega' = \Omega$, and

$$\Delta' = \Delta \cup \{\mathsf{cons}(\mathsf{Nat}, \mathsf{List}) \xrightarrow{0} \mathsf{List}, \mathsf{nil} \xrightarrow{0} \mathsf{List}\}.$$

$\mathcal{A}'$ is deterministic (but not strongly), simple, and not complete, and $\mathcal{L}(\mathcal{A}')$ is the set of subsets of the regular tree language L of Lisp-like lists of natural numbers. Indeed, successful runs can now be defined on generalized tree sets g such that a term in L is labeled by 0 or 1.

Ex. 46.3 Let $E = \{0, 1\}^2$, $\mathcal{F} = \{\mathsf{cons}(,), s(), \mathsf{nil}, 0\}$, and let $\mathcal{A} = (Q, \Delta, \Omega)$ be defined by $Q = \{\mathsf{Nat}, \mathsf{Nat}', \mathsf{List}, \mathsf{Term}\}$, $\Omega = 2^Q$, and Δ is the following set of rules:

$$\begin{array}{rcl}
0 \xrightarrow{(0,0)} \mathsf{Nat} \quad ; & 0 \xrightarrow{(1,0)} \mathsf{Nat}' & ; \; s(\mathsf{Nat}) \xrightarrow{(0,0)} \mathsf{Nat} \\
s(\mathsf{Nat}) \xrightarrow{(1,0)} \mathsf{Nat}' \; ; & s(\mathsf{Nat}') \xrightarrow{(0,0)} \mathsf{Nat} & ; \; s(\mathsf{Nat}') \xrightarrow{(1,0)} \mathsf{Nat}' \\
\mathsf{nil} \xrightarrow{(0,1)} \mathsf{List} \quad ; & \mathsf{cons}(\mathsf{Nat}', \mathsf{List}) \xrightarrow{(0,1)} \mathsf{List} \; ; & \\
 & s(q) \xrightarrow{(0,0)} \mathsf{Term} \quad \forall q \neq \mathsf{Nat} & \\
 & \mathsf{cons}(q, q') \xrightarrow{(0,0)} \mathsf{Term} \quad \forall (q, q') \neq (\mathsf{Nat}', \mathsf{List}) &
\end{array}$$

$\mathcal{A}$ is deterministic, simple, and not complete, and $\mathcal{L}(\mathcal{A})$ is the set of 2-tuples of tree languages (N', L') where N' is a subset of the regular tree language of natural numbers and L' is the set of Lisp-like lists of natural numbers over N'.

Let us remark that the set N' may be non-regular. For instance, one can define a run on a characteristic generalized tree set g_p of Lisp-like lists of prime numbers. The generalized tree set g_p is such that $g_p(t) = (1, 0)$ when t is a (code of a) prime number.

In the previous examples, we only consider simple generalized tree set automata. Moreover all runs are successful runs. The following examples are non-simple generalized tree set automata in order to make clear the interest of acceptance conditions. For this, compare the sets of generalized tree sets obtained in examples 46.3 and 47 and note that with acceptance conditions, we can express that a set is non empty.

Example 47. *Example 46.3 continued*

Let $E = \{0, 1\}^2$, $\mathcal{F} = \{\mathsf{cons}(,), \mathsf{nil}, s(), 0\}$, and let $\mathcal{A}' = (Q', \Delta', \Omega')$ be defined by $Q' = Q$, $\Delta' = \Delta$, and $\Omega' = \{\omega \in 2^Q \mid \mathsf{Nat}' \in \omega\}$. $\mathcal{A}'$ is deterministic, not simple, and not complete, and $\mathcal{L}(\mathcal{A}')$ is the set of 2-tuples of tree languages (N', L') where N' is a subset of the regular tree language of natural numbers and L' is the set of Lisp-like lists of natural numbers over N', and $N' \neq \emptyset$. Indeed, for a successful r on g, there must be a term t such that $r(t) = \mathsf{Nat}'$ therefore, there must be a term t labelled by $(1, 0)$, henceforth $N' \neq \emptyset$.

5.2.3 Hierarchy of GTSA-recognizable languages

Let us define:

- $\mathcal{R}_{\mathrm{GTS}}$, the class of languages recognizable by GTSA,
- $\mathcal{R}_{\mathrm{DGTS}}$, the class of languages recognizable by deterministic GTSA,
- $\mathcal{R}_{\mathrm{SGTS}}$, the class of languages recognizable by Simple GTSA.

The three classes defined above are proved to be different. They are also closely related to classes of languages defined from the set constraint theory point of view.

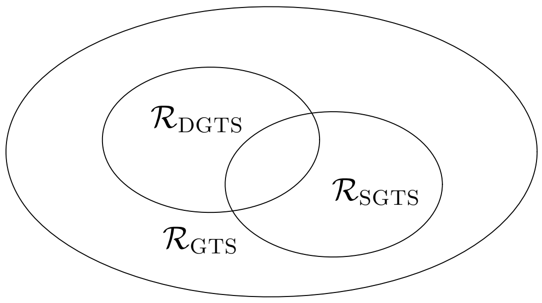

$\mathcal{R}_{\mathrm{DGTS}}$
$\mathcal{R}_{\mathrm{SGTS}}$
$\mathcal{R}_{\mathrm{GTS}}$

Figure 5.5: Classes of GTSA-recognizable languages

Classes of GTSA-recognizable languages have also different closure properties. We will prove in Section 5.3.1 that $\mathcal{R}_{\mathrm{SGTS}}$ and the entire class $\mathcal{R}_{\mathrm{GTS}}$ are closed under union, intersection, projection and cylindrification; $\mathcal{R}_{\mathrm{DGTS}}$ is closed under complementation and intersection.

We propose three examples that illustrate the differences between the three classes. First, $\mathcal{R}_{\mathrm{DGTS}}$ is not a subset of $\mathcal{R}_{\mathrm{SGTS}}$.

Example 48. Let $E = \{0, 1\}$, $\mathcal{F} = \{f, a\}$ where a is a constant and f is unary. Let us consider the deterministic but non-simple GTSA $\mathcal{A}_1 = (Q_1, \Delta_1, \Omega_1)$ where Δ_1 is:

$$\begin{array}{ll} a \xrightarrow{0} q_0, & a \xrightarrow{1} q_1, \\ f(q_0) \xrightarrow{0} q_0, & f(q_1) \xrightarrow{0} q_0, \\ f(q_0) \xrightarrow{1} q_1, & f(q_1) \xrightarrow{1} q_0. \end{array}$$

and $\Omega_1 = \{\{q_0, q_1\}, \{q_1\}\}$. Let us proove that

$$\mathcal{L}(\mathcal{A}_1) = \{L \mid L \neq \emptyset\}$$

is not in $\mathcal{R}_{\mathrm{SGTS}}$.

Assume that there exists a simple GTSA $\mathcal{A}_s$ with n states such that $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_s)$. Hence, $\mathcal{A}_s$ recognizes also each one of the singleton sets $\{f^i(a)\}$ for $i > 0$. Let us consider some i greater than $n+1$, we can deduce that a run r on the GTS g associated with $\{f^i(a)\}$ maps two terms $f^k(a)$ and $f^l(a)$, $k < l < i$ to the same state. We have $g(t) = 0$ for every term $t \trianglelefteq f^l(a)$ and r "loops" between $f^k(a)$ and $f^l(a)$. Therefore, one can build another run r_0 on a GTS g_0 such that $g_0(t) = 0$ for each $t \in T(\mathcal{F})$. Since $\mathcal{A}_s$ is simple, and since the range of r_0 is a subset of the range of r, g_0 is recognized, hence the empty set is recognized which contradicts the hypothesis.

Basically, using simple GTSA it is not possible to enforce a state to be assumed somewhere by every run. Consequently, it is not possible to express global properties of generalized tree languages such as non-emptiness.

Second, $\mathcal{R}_{\mathrm{SGTS}}$ is not a subset of $\mathcal{R}_{\mathrm{DGTS}}$.

Example 49. Let us consider the non-deterministic but simple GTSA $\mathcal{A}_2 = (Q_2, \Delta_2, \Omega_2)$ where Δ_2 is:

$$\begin{array}{ll}
a \xrightarrow{0} q_f \mid q_h, & a \xrightarrow{1} q_f \mid q_h, \\
f(q_f) \xrightarrow{1} q_f \mid q_h, & h(q_h) \xrightarrow{1} q_f \mid q_h, \\
f(q_h) \xrightarrow{0} q_f \mid q_h, & h(q_f) \xrightarrow{0} q_f \mid q_h,
\end{array}$$

and $\Omega = 2^{\{q_f, q_h\}}$. It is easy to prove

$$\mathcal{L}(\mathcal{A}_2) = \{L \mid \forall t \; f(t) \in L \Leftrightarrow h(t) \notin L\}.$$

The proof that no deterministic GTSA recognizes $\mathcal{L}(\mathcal{A}_2)$ is left to the reader.

We terminate with an example of a non-deterministic and non-simple generalized tree set automaton. This example will be used in the proof of Proposition 36.

Example 50. Let $\mathcal{A} = (Q, \Delta, \Omega)$ be defined by $Q = \{q, q'\}$, $\Omega = \{Q\}$, and Δ is the following set of rules:

$$\begin{array}{llll}
a \xrightarrow{1} q & ; a \xrightarrow{1} q' & ; a \xrightarrow{0} q' & ; f(q) \xrightarrow{1} q \; ; \\
f(q') \xrightarrow{0} q' & ; f(q') \xrightarrow{1} q' & ; f(q') \xrightarrow{1} q \; ; &
\end{array}$$

The proof that $\mathcal{A}$ is not deterministic, not simple, and not complete, and $\mathcal{L}(\mathcal{A}) = \{L \subseteq T(\mathcal{F}) \mid \exists t \in T(\mathcal{F}) \; ((t \in L) \wedge (\forall t' \in T(\mathcal{F}) \; (t \trianglelefteq t') \Rightarrow (t' \in L)))\}$ is left as an exercise to the reader.

5.2.4 Regular generalized tree sets, regular runs

As we mentioned it in Example 46.3, the set recognized by a GTSA may contain GTS corresponding to non-regular languages. But regularity is of major interest for practical reasons because it implies a GTS or a language to be finitely defined.

A generalized tree set $g \in \mathcal{G}_E$ is ***regular*** if there exist a finite set R, a map $\alpha : T(\mathcal{F}) \to R$, and a map $\beta : R \to E$ satisfying the following two properties.

1. $g = \alpha\beta$ (*i.e.* $g = \beta \circ \alpha$),
2. α is closed under contexts, i.e. for all context c and terms t_1, t_2, we have

$$(\alpha(t_1) = \alpha(t_2)) \Rightarrow (\alpha(c[t_1]) = \alpha(c[t_2]))$$

In the case $E = \{0,1\}^n$, regular generalized tree sets will correspond to n-tuples of regular tree languages.

Although the definition of regularity leads to the definition of regular run — because a run can be considered as a generalized tree set in $\mathcal{G}_Q$, we use stronger conditions for a run to be regular. Indeed, regularity of generalized tree sets and regularity of runs do not correspond in general. For instance, one could define regular runs on non-regular generalized tree sets in the case of non-strongly deterministic generalized tree set automata, and one could define non-regular runs on regular generalized tree sets in the case of non-deterministic generalized tree set automata.

Therefore, we only consider regular runs on regular generalized tree sets:

> A run r on a generalized tree set g is regular if $r \uparrow g \in \mathcal{G}_{E\times\mathcal{Q}}$ is regular. Consequently, r and g are regular generalized tree sets.

Proposition 33. *Let $\mathcal{A}$ be a generalized tree set automaton, if g is a regular generalized tree set in $\mathcal{L}(\mathcal{A})$ then there exists a regular $\mathcal{A}$-run on g.*

Proof. Consider a generalized tree set automaton $\mathcal{A} = (Q, \Delta, \Omega)$ over E and a regular generalized tree set g in $\mathcal{L}(\mathcal{A})$ and let r be a successful run on g. Let L be a tree language closed under the subterm relation and such that $\mathcal{F}_0 \subseteq L$ and $r(L) = r(T(\mathcal{F}))$. The generalized tree set g is regular, therefore there exist a finite set R, a mapping $\alpha : T(\mathcal{F}) \to R$ closed under context and a mapping $\beta : R \to E$ such that $g = \alpha\beta$. We now define a regular run r' on g.

Let $L_\star = L \cup \{\star\}$ where $\star$ is a new constant symbol and let ϕ be the mapping from $T(\mathcal{F})$ into $Q \times R \times L_\star$ defined by $\phi(t) = (r(t), \alpha(t), u)$ where $u = t$ if $t \in L$ and $u = \star$ otherwise. Hence $R' = \phi(T(\mathcal{F}))$ is a finite set because $R' \subseteq Q \times R \times L_\star$. For each ρ in R', let us fix $t_\rho \in T(\mathcal{F})$ such that $\phi(t_\rho) = \rho$.

The run r' is now (regularly) defined via two mappings α' and β'. Let β' be the projection from $Q \times R \times L_\star$ into Q and let $\alpha' : T(\mathcal{F}) \to R'$ be inductively defined by :

$$\forall a \in \mathcal{F}_0\ \alpha'(a) = \phi(a);$$

and

$$\forall f \in \mathcal{F}_p \forall t_1, \ldots, t_p \in T(\mathcal{F})$$

$$\alpha'(f(t_1, \ldots, t_p)) = \phi(f(t_{\alpha'(t_1)}, \ldots, t_{\alpha'(t_p)})).$$

Let $r' = \alpha'\beta'$. First we can easily prove by induction that $\forall t \in L\ \alpha'(t) = \phi(t)$ and deduce that $\forall t \in L\ r'(t) = r(t)$. Thus r' and r coincide on L. It remains to prove that (1) the mapping α' is closed under context, (2) r' is a run on g and (3) r' is a successful run.

(1) From the definition of α' we can easily derive that the mapping α' is closed under context.

(2) We prove that the mapping $r' = \alpha'\beta'$ is a run on g, that is if $t = f(t_1, \dots, t_p)$ then $(r'(t_1), \dots, r'(t_p), f, g(t), r'(t)) \in \Delta$.

Let us consider a term $t = f(t_1, \dots, t_p)$. From the definitions of α', β', and r', we get $r'(t) = r(t')$ with $t' = f(t_{\alpha'(t_1)}, \dots, t_{\alpha'(t_p)})$. The map r is a run on g, hence $(r(t_{\alpha'(t_1)}), \dots, r(t_{\alpha'(t_p)}), f, g(t'), r(t')) \in \Delta$, and thus it suffices to prove that $g(t) = g(t')$ and, for all i, $r'(t_i) = r(t_{\alpha'(t_i)})$.

Let $i \in \{1, \dots, p\}$, $r'(t_i) = \beta'(\alpha'(t_i))$ by definition of r'. By definition of $t_{\alpha'(t_i)}$, $\alpha'(t_i) = \phi(t_{\alpha'(t_i)})$, therefore $r'(t_i) = \beta'(\phi(t_{\alpha'(t_i)}))$. Now, using the definitions of ϕ and β', we get $r'(t_i) = r(t_{\alpha'(t_i)})$.

In order to prove that $g(t) = g(t')$, we prove that $\alpha(t) = \alpha(t')$. Let π be the projection from R' into R. We have $\alpha(t') = \pi(\phi(t'))$ by definition of ϕ and π. We have $\alpha(t') = \pi(\alpha'(t))$ using definitions of t' and α'. Now $\alpha(t') = \pi(\phi(t_{\alpha'(t)}))$ because $\phi(t_{\alpha'(t)}) = \alpha'(t)$ by definition of $t_{\alpha'(t)}$. And then $\alpha(t') = \alpha(t_{\alpha'(t)})$ by definition of π and ϕ. Therefore it remains to prove that $\alpha(t_{\alpha'(t)}) = \alpha(t)$. The proof is by induction on the structure of terms.

If $t \in \mathcal{F}_0$ then $t_{\alpha'(t)} = t$, so the property holds (note that this property holds for all $t \in L$). Let us suppose that $t = f(t_1, \dots, t_p)$ and $\alpha(t_{\alpha'(t_i)}) = \alpha(t_i)\ \forall i \in \{1, \dots, p\}$. First, using induction hypothesis and closure under context of α, we get

$$\alpha(f(t_1, \dots, t_p)) \quad = \quad \alpha(f(t_{\alpha'(t_1)}, \dots, t_{\alpha'(t_p)}))$$

Therefore,

$$\begin{aligned}
\alpha(f(t_1, \dots, t_p)) &= \alpha(f(t_{\alpha'(t_1)}, \dots, t_{\alpha'(t_p)})) \\
&= \pi(\phi(f(t_{\alpha'(t_1)}, \dots, t_{\alpha'(t_p)}))) \text{ (def. of } \phi \text{ and } \pi) \\
&= \pi(\alpha'(f(t_1, \dots, t_p))) \text{ (def. of } \alpha') \\
&= \pi(\phi(t_{\alpha'(f(t_1, \dots, t_p))})) \text{ (def. of } t_{\alpha'(f(t_1, \dots, t_p))}) \\
&= \alpha(t_{\alpha'(f(t_1, \dots, t_p))}) \text{ (def. of } \phi \text{ and } \pi).
\end{aligned}$$

(3) We have $r'(T(\mathcal{F})) = r'(L) = r(L) = r(T(\mathcal{F}))$ using the definition of r', the definition of L, and the equality $r'(L) = r(L)$. The run r is a successful run. Consequently r' is a successful run.

□

Proposition 34. *A non-empty recognizable set of generalized tree sets contains a regular generalized tree set.*

Proof. Let us consider a generalized tree set automaton $\mathcal{A}$ and a successful run r on a generalized tree set g. There exists a tree language closed under the subterm relation F such that $r(F) = r(T(\mathcal{F}))$. We define a regular run rr on a regular generalized tree set gg in the following way.

The run rr coincides with r on $F : \forall t \in F$, $rr(t) = r(t)$ and $gg(t) = g(t)$. The runs rr and gg are inductively defined on $T(\mathcal{F}) \setminus F$: given $q_1, \ldots, q_p$ in $r(T(\mathcal{F}))$, let us fix a rule $f(q_1, \ldots, q_p) \xrightarrow{l} q$ such that $q \in r(T(\mathcal{F}))$. The rule exists since r is a run. Therefore, $\forall t = f(t_1, \ldots, t_p) \notin F$ such that $rr(t_i) = q_i$ for all $i \leq p$, we define $rr(t) = q$ and $gg(t) = l$, following the fixed rule $f(q_1, \ldots, q_p) \xrightarrow{l} q$.

□

From the preceding, we can also deduce that a finite and recognizable set of generalized tree sets only contains regular generalized tree sets.

5.3 Closure and decision properties

5.3.1 Closure properties

This section is dedicated to the study of classical closure properties on GTSA-recognizable languages. For all positive results — union, intersection, projection, cylindrification — the proofs are constructive. We show that the class of recognizable sets of generalized tree sets is not closed under complementation and that non-determinism cannot be reduced for generalized tree set automata.

Set operations on sets of GTS have to be distinguished from set operations on sets of terms. In particular, in the case where $E = \{0,1\}^n$, if G_1 and G_2 are sets of GTS in $\mathcal{G}_E$, $G_1 = (L_1^1, \ldots, L_n^1)$ and $G_2 = (L_1^2, \ldots, L_n^2)$, then $G_1 \cup G_2$ contains all GTS in G_1 and G_2. This is clearly different from the set $(L_1^1 \cup L_1^2, \ldots, L_n^1 \cup L_n^2)$.

Proposition 35. *The class $\mathcal{R}_{\mathrm{GTS}}$ is closed under intersection and union, i.e. if G_1, $G_2 \subseteq \mathcal{G}_E$ are recognizable, then $G_1 \cup G_2$ and $G_1 \cap G_2$ are recognizable.*

This proof is an easy modification of the classical proof of closure properties for tree automata, see Chapter 1.

Proof. Let $\mathcal{A}_1 = (Q_1, \Delta_1, \Omega_1)$ and $\mathcal{A}_2 = (Q_2, \Delta_2, \Omega_2)$ be two generalized tree set automata over E. Without loss of generality we suppose that $Q_1 \cap Q_2 = \emptyset$.

Let $\mathcal{A} = (Q, \Delta, \Omega)$ with $Q = Q_1 \cup Q_2$, $\Delta = \Delta_1 \cup \Delta_2$, and $\Omega = \Omega_1 \cup \Omega_2$. It is immediate that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_2)$.

We denote by π_1 and π_2 the projections from $Q_1 \times Q_2$ into respectively Q_1 and Q_2. Let $\mathcal{A}' = (Q', \Delta', \Omega')$ with $Q' = Q_1 \times Q_2$, Δ' is defined by

$$(f(q_1, \ldots, q_p) \xrightarrow{l} q \in \Delta') \Leftrightarrow (\forall i \in \{1,2\}\ f(\pi_i(q_1), \ldots, \pi_i(q_p)) \xrightarrow{l} \pi_i(q) \in \Delta_i)\ ,$$

where $q_1, \ldots, q_p, q \in Q'$, $f \in \mathcal{F}_p$, $l \in E$, and Ω' is defined by

$$\Omega' = \{\omega \in 2^{Q'} \mid \pi_i(\omega) \in \Omega_i\ ,\ i \in \{1,2\}\}.$$

One can easily verify that $\mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$.

□

Let us remark that the previous constructions also prove that the class $\mathcal{R}_{\mathrm{SGTS}}$ is closed under union and intersection.

The class languages recognizable by deterministic generalized tree set automata is closed under complementation. But, this property is false in the general case of GTSA-recognizable languages.

Proposition 36. *(a) Let $\mathcal{A}$ be a generalized tree set automaton, there exists a complete generalized tree set automaton $\mathcal{A}_c$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_c)$.*

(b) If $\mathcal{A}_{cd}$ is a deterministic and complete generalized tree set automaton, there exists a generalized tree set automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}') = \mathcal{G}_E - \mathcal{L}(\mathcal{A}_{cd})$.

(c) The class of GTSA-recognizable languages is not closed under complementation.

(d) Non-determinism can not be reduced for generalized tree set automata.

Proof. (a) Let $\mathcal{A} = (Q, \Delta, \Omega)$ be a generalized tree set automaton over E and let q' be a new state, i.e. $q' \notin Q$. Let $\mathcal{A}_c = (Q_c, \Delta_c, \Omega_c)$ be defined by $Q_c = Q \cup \{q'\}$, $\Omega_c = \Omega$, and

$$\begin{aligned}\Delta_c = \Delta \cup \{(q_1, \dots, q_p, f, l, q') \mid \quad & \{(q_1, \dots, q_p, f, l)\} \times Q \cap \Delta = \emptyset; \\ & q_1, \dots, q_p \in Q_c, f \in \mathcal{F}_p, l \in E\}.\end{aligned}$$

$\mathcal{A}_c$ is complete and $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_c)$. Note that $\mathcal{A}_c$ is simple if $\mathcal{A}$ is simple.

(b) $\mathcal{A}_{cd} = (Q, \Delta, \Omega)$ be a deterministic and complete generalized tree set automaton over E. The automaton $\mathcal{A}' = (Q', \Delta', \Omega')$ with $Q' = Q$, $\Delta' = \Delta$, and $\Omega' = 2^Q - \Omega$ recognizes the set $\mathcal{G}_E - \mathcal{L}(\mathcal{A}_{cd})$.

(c) $E = \{0, 1\}$, $\mathcal{F} = \{c, a\}$ where a is a constant and c is of arity 1. Let $G = \{g \in \mathcal{G}_{\{0,1\}^n} \mid \exists t \in T(\mathcal{F})\ ((g(t) = 1) \wedge (\forall t' \in T(\mathcal{F})\ (t \trianglelefteq t') \Rightarrow (g(t') = 1)))\}$. Clearly, G is recognizable by a non deterministic GTSA (see Example 50). Let $\overline{G} = \mathcal{G}_{\{0,1\}^n} - G$, we have $\overline{G} = \{g \in \mathcal{G}_{\{0,1\}^n} \mid \forall t \in T(\mathcal{F})\ \exists t' \in T(\mathcal{F})\ (t \trianglelefteq t') \wedge (g(t') = 0)\}$ and $\overline{G}$ is not recognizable. Let us suppose that $\overline{G}$ is recognized by an automaton $\mathcal{A} = (Q, \Delta, \Omega)$ with $\mathsf{Card}(Q) = k - 2$ and let us consider the generalized tree set g defined by: $g(c^i(a)) = 0$ if $i = k \times z$ for some integer z, and $g(c^i(a)) = 1$ otherwise. The generalized tree set g is in $\overline{G}$ and we consider a successful run r on g. We have $r(T(\mathcal{F})) = \omega \in \Omega$ therefore there exists some integer n such that $r(\{g(c^i(a)) \mid i \leq n\}) = \omega$. Moreover we can suppose that n is a multiple of k. As $\mathsf{Card}(Q) = k - 2$ there are two terms u and v in the set $\{c^i(a) \mid n+1 \leq i \leq n+k-1\}$ such that $r(u) = r(v)$. Note that by hypothesis, for all i such that $n+1 \leq i \leq n+k+1$, $g(c^i(a)) = 1$. Consequently, a successful run g' could be defined from g on the generalized tree set g' defined by $g'(t) = g(t)$ if $t = c^i(a)$ when $i \leq n$, and $g'(t) = 1$ otherwise. This leads to a contradiction because $g' \notin \overline{G}$.

(d) This result is a consequence of (b) and (c).

□

We will now prove the closure under projection and cylindrification. We will first prove a stronger lemma.

Lemma 8. *Let $G \subseteq \mathcal{G}_{E_1}$ be a GTSA-recognizable language and let $R \subseteq E_1 \times E_2$. The set $R(G) = \{g' \in \mathcal{G}_{E_2} \mid \exists g \in G\ \forall t \in T(\mathcal{F})\ (g(t), g'(t)) \in R\}$ is recognizable.*

Proof. Let $\mathcal{A} = (Q, \Delta, \Omega)$ such that $\mathcal{L}(\mathcal{A}) = G$. Let $\mathcal{A}' = (Q', \Delta', \Omega')$ where $Q' = Q$, $\Delta' = \{f(q_1, \ldots, q_p) \xrightarrow{l'} q \mid \exists l \in E_1\ f(q_1, \ldots, q_p) \xrightarrow{l} q \in \Delta \text{ and } (l, l') \in R\}$ and $\Omega' = \Omega$. We prove that $R(G) = \mathcal{L}(\mathcal{A}')$.

$\supseteq$ Let $g' \in \mathcal{L}(\mathcal{A}')$ and let r' be a successful run on g'. We construct a generalized tree set g such that for all $t \in T(\mathcal{F})$, $(g(t), g'(t)) \in R$ and such that r' is also a successful $\mathcal{A}$-run on g.

Let a be a constant. According to the definition of Δ', $a \xrightarrow{g'(a)} r'(a) \in \Delta'$ implies that there exists l_a such that $(l_a, g'(a)) \in R$ and $a \xrightarrow{l_a} r'(a) \in \Delta$. So let $g(a) = l_a$.

Let $t = f(t_1, \ldots, t_p)$ with $\forall i\ r'(t_i) = q_i$. There exists a rule $f(q_1, \ldots, q_p) \xrightarrow{g'(t)} r'(t)$ in Δ' because r' is a run and again, from the definition of Δ', there exists $l_t \in E_1$ such that $f(q_1, \ldots, q_p) \xrightarrow{l_t} r'(t)$ in Δ with $(l_t(t), g'(t)) \in R$. So, we define $g(t) = l_t$. Clearly, g is a generalized tree set and r' is a successful run on g and for all $t \in T(\mathcal{F})$, $(g(t), g'(t)) \in R$ by construction.

$\subseteq$ Let $g' \in R(G)$ and let $g \in G$ such that $\forall t \in T(\mathcal{F})\ (g(t), g'(t)) \in R$. One can easily prove that any successful $\mathcal{A}$-run on g is also a successful $\mathcal{A}'$-run on g'.

□

Corollary 7. *(a) The class of GTSA-recognizable languages is closed under projection and cylindrification.*

(b) Let $G \subseteq \mathcal{G}_E$ and $G' \subseteq \mathcal{G}_{E'}$ be two GTSA-recognizable languages. The set $G \uparrow G' = \{g \uparrow g' \mid g \in G,\ g' \in G'\}$ is a GTSA-recognizable language in $\mathcal{G}_{E \times E'}$.

Proof. (a) The case of projection is an immediate consequence of Lemma 8 using $E_1 = E \times E'$, $E_2 = E$, and $R = \pi$ where π is the projection from $E \times E'$ into E. The case of cylindrification is proved in a similar way.

(b) Consequence of (a) and of Proposition 35 because $G \uparrow G' = \pi_1^{-1}(G) \cap \pi_2^{-1}(G')$ where π_1^{-1} (respectively π_2^{-1}) is the inverse projection from E to $E \times E'$ (respectively from E' to $E \times E'$).

Let us remark that the construction preserves simplicity, so $\mathcal{R}_{\mathrm{SGTS}}$ is closed under projection and cylindrification.

□

We now consider the case $E = \{0, 1\}^n$ and we give two propositions without proof. Proposition 37 can easily be deduced from Corollary 7. The proof of Proposition 38 would be an extension of the constructions made in Examples 46.1 and 46.2.

Proposition 37. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two generalized tree set automata over $\{0, 1\}^n$.*

(a) $\{(L_1 \cup L_1', \ldots, L_n \cup L_n') \mid (L_1, \ldots, L_n) \in \mathcal{L}(\mathcal{A})$ and $(L_1', \ldots, L_n') \in \mathcal{L}(\mathcal{A}')\}$ is recognizable.

(b) $\{(L_1 \cap L'_1, \dots, L_n \cap L'_n) \mid (L_1, \dots, L_n) \in \mathcal{L}(\mathcal{A})$ *and* $(L'_1, \dots, L'_n) \in \mathcal{L}(\mathcal{A}')\}$ *is recognizable.*

(c) $\{(\overline{L_1}, \dots, \overline{L_n}) \mid (L_1, \dots, L_n) \in \mathcal{L}(\mathcal{A})\}$ *is recognizable, where* $\overline{L_i} = T(\mathcal{F}) - L_i$, $\forall i$.

Proposition 38. *Let* $E = \{0,1\}^n$ *and let* $(F_1, \dots, F_n)$ *be a* n*-tuple of regular tree languages. There exist deterministic simple generalized tree set automata* $\mathcal{A}$ *,* $\mathcal{A}'$*, and* $\mathcal{A}''$ *such that*

- $\mathcal{L}(\mathcal{A}) = \{(F_1, \dots, F_n)\}$;
- $\mathcal{L}(\mathcal{A}') = \{(L_1, \dots, L_n) \mid L_1 \subseteq F_1, \dots, L_n \subseteq F_n\}$;
- $\mathcal{L}(\mathcal{A}'') = \{(L_1, \dots, L_n) \mid F_1 \subseteq L_1, \dots, F_n \subseteq L_n\}$.

5.3.2 Emptiness property

Theorem 41. *The emptiness property is decidable in the class of generalized tree set automata. Given a generalized tree set automaton* $\mathcal{A}$*, it is decidable whether* $\mathcal{L}(\mathcal{A}) = \emptyset$.

Labels of the generalized tree sets are meaningless for the emptiness decision thus we consider "label-free" generalized tree set automata. Briefly, the transition relation of a "label-free" generalized tree set automata is a relation $\Delta \subseteq \cup_p Q^p \times \mathcal{F}_p \times Q$.

The emptiness decision algorithm for simple generalized tree set automata is straightforward. Indeed, Let ω be a subset of Q and let $\mathsf{COND}(\omega)$ be the following condition:

$$\forall p \; \forall f \in \mathcal{F}_p \; \forall q_1, \dots, q_p \in \omega \; \exists q \in \omega \quad (q_1, \dots, q_p, f, q) \in \Delta$$

We easily prove that there exists a set ω satisfying $\mathsf{COND}(\omega)$ if and only if there exists an $\mathcal{A}$-run. Therefore, the emptiness problem for simple generalized tree set automata is decidable because 2^Q is finite and $\mathsf{COND}(\omega)$ is decidable. Decidability of the emptiness problem for simple generalized tree set automata is NP-complete (see Prop. 39).

The proof is more intricate in the general case, and it is not given in this book. Without the property of simple GTSA, we have to deal with a reachability problem of a set of states since we have to check that there exists $\omega \in \Omega$ and a run r such that r assumes exactly all the states in ω.

We conclude this section with a complexity result of the emptiness problem in the class of generalized tree set automata.

Let us consider a GTSA $\mathcal{A}$ with n states. The proof shows that one has to consider at most all tree languages closed under the subterm relation of size smaller than $B(\mathcal{A})$, a polynomial in n, in order to decide emptiness for $\mathcal{A}$. Let us remark that the polynomial bound $B(\mathcal{A})$ can be computed.

Proposition 39. *The emptiness problem in the class of (simple) generalized tree set automata is NP-complete.*

Proof. Let $\mathcal{A} = (Q, \Delta, \Omega)$ be a generalized tree set automaton over E. Let $n = \mathsf{Card}(Q)$.

We first give a non-deterministic and polynomial algorithm for deciding emptiness: (1) take a tree language F closed under the subterm relation of size smaller than $B(\mathcal{A})$; (2) take a run r on F; (3) compute $r(F)$; (4) check whether $r(F) = \omega$ is a member of Ω; (5) check whether ω satisfies $\mathsf{COND}(\omega)$.

From Theorem 41, this algorithm is correct and complete. Moreover, this algorithm is polynomial in n since the size of F is polynomial in n: step (2) consists in labeling the nodes of F with states following the rules of the automaton – so there is a polynomial number of states, step (3) consists in collecting the states; step (4) is polynomial and non-deterministic and finally, step (5) is polynomial.

We reduce the satisfiability problem of boolean expressions into the emptiness problem for generalized tree set automata. We first build a generalized tree set automaton $\mathcal{A}$ such that $L(\mathcal{A})$ is the set of (codes of) satisfiable boolean expressions over n variables $\{x_1, \ldots, x_n\}$.

Let $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$ where $\mathcal{F}_0 = \{x_1, \ldots, x_n\}$, $\mathcal{F}_1 = \{\neg\}$, and $\mathcal{F}_2 = \{\wedge, \vee\}$. A boolean expression is a term of $T(\mathcal{F})$. Let $\mathsf{Bool} = \{0, 1\}$ be the set of boolean values. Let $\mathcal{A} = (Q, \Delta, \Omega)$, be a generalized tree set automaton such that $Q = \{q_0, q_1\}$, $\Omega = 2^Q$ and Δ is the following set of rules:

$$\begin{aligned}
x_j \xrightarrow{i} q_i &\text{ where } j \in \{1, \ldots, n\} \text{ and } i \in \mathsf{Bool} \\
\neg(q_i) \xrightarrow{\neg i} q_{\neg i} &\text{ where } i \in \mathsf{Bool} \\
\vee(q_{i_1}, q_{i_2}) \xrightarrow{i_1 \vee i_2} q_{i_1 \vee i_2} &\text{ where } i_1, i_2 \in \mathsf{Bool} \\
\wedge(q_{i_1}, q_{i_2}) \xrightarrow{i_1 \wedge i_2} q_{i_1 \wedge i_2} &\text{ where } i_1, i_2 \in \mathsf{Bool}
\end{aligned}$$

One can easily prove that $L(\mathcal{A}) = \{L_v \mid v \text{ \textit{is a valuation of} } \{x_1, \ldots, x_n\}\}$ where $L_v = \{t \mid t \text{ \textit{is a boolean expression which is true under} } v\}$. Now, we can derive an algorithm for the satisfiability of any boolean expression e: build $\mathcal{A}_e$ a strongly deterministic generalized tree set automaton such that $\mathcal{L}(\mathcal{A})$ is the set $\{L \mid e \in L\}$; build $\mathcal{A}_e \cap \mathcal{A}$ and decide emptiness.

We get then the reduction because $\mathcal{A}_e \cap \mathcal{A}$ is empty if and only if e is not satisfiable.

Now, it remains to prove that the reduction is polynomial. The size of $\mathcal{A}$ is $2 * n + 10$. The size of $\mathcal{A}_e$ is the length of e plus a constant. So we got the result. □

5.3.3 Other decision results

Proposition 40. *The inclusion problem and the equivalence problem for deterministic generalized tree set automata are decidable.*

Proof. These results are a consequence of the closure properties under intersection and complementation (Propositions 35, 36), and the decidability of the emptiness property (Theorem 41). □

Proposition 41. *Let $\mathcal{A}$ be a generalized tree set automaton. It is decidable whether or not $\mathcal{L}(\mathcal{A})$ is a singleton set.*

Proof. Let $\mathcal{A}$ be a generalized tree set automaton. First it is decidable whether $\mathcal{L}(\mathcal{A})$ is empty or not (Theorem 41). Second if $\mathcal{L}(\mathcal{A})$ is non empty then a regular generalized tree set g in $\mathcal{L}(\mathcal{A})$ can be constructed (see the proof of Theorem 41). Construct the strongly deterministic generalized tree set automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}')$ is a singleton set reduced to the generalized tree set g. Finally, build $\mathcal{A} \cap \overline{\mathcal{A}}'$ to decide the equivalence of $\mathcal{A}$ and $\mathcal{A}'$. Note that we can build $\overline{\mathcal{A}}'$, since $\mathcal{A}'$ is deterministic (see Proposition 36). □

Proposition 42. *Let $L = (L_1, \dots, L_n)$ be a tuple of regular tree language and let $\mathcal{A}$ be a generalized tree set automaton over $\{0,1\}^n$. It is decidable whether $L \in \mathcal{L}(\mathcal{A})$.*

Proof. This result just follows from closure under intersection and emptiness decidability.

First construct a (strongly deterministic) generalized tree set automaton $\mathcal{A}_L$ such that $L(\mathcal{A})$ is reduced to the singleton set $\{L\}$. Second, construct $\mathcal{A} \cap \mathcal{A}_L$ and decide whether $L(\mathcal{A} \cap \mathcal{A}_L)$ is empty or not. □

Proposition 43. *Given a generalized tree set automaton over $E = \{0, 1, \}^n$ and $I \subseteq \{1, \dots, n\}$. The following two problems are decidable:*

1. *It is decidable whether or not there exists $(L_1, \dots, L_n)$ in $\mathcal{L}(\mathcal{A})$ such that all the L_i are finite for $i \in I$.*

2. *Let $x_1 \dots, x_n$ be natural numbers. It is decidable whether or not there exists $(L_1, \dots, L_n)$ in $\mathcal{L}(\mathcal{A})$ such that $\mathsf{Card}(L_i) = x_i$ for each $i \in I$.*

The proof is technical and not given in this book. It relies on a lemma of the emptiness decision proof.

5.4 Applications to set constraints

In this section, we consider the satisfiability problem for systems of set constraints. We show a decision algorithm using generalized tree set automata.

5.4.1 Definitions

Let $\mathcal{F}$ be a finite and non-empty set of function symbols. Let $\mathcal{X}$ be a set of variables. We consider special symbols $\top$, $\bot$, $\sim$, $\cup$, $\cap$ of respective arities 0, 0, 1, 2, 2. A *set expression* is a term in $T_{\mathcal{F}'}(\mathcal{X})$ where $\mathcal{F}' = \mathcal{F} \cup \{\top, \bot, \sim, \cup, \cap\}$.

A set constraint is either a *positive* set constraint of the form $e \subseteq e'$ or a *negative* set constraint of the form $e \not\subseteq e'$ (or $\neg(e \subseteq e')$) where e and e' are set expressions, and a system of set constraints is defined by $\bigwedge_{i=1}^{k} SC_i$ where the SC_i are set constraints.

An interpretation $\mathcal{I}$ is a mapping from $\mathcal{X}$ into $2^{T(\mathcal{F})}$. It can immediately be extended to set expressions in the following way:

$$\begin{aligned}
\mathcal{I}(\top) &= T(\mathcal{F});\\
\mathcal{I}(\bot) &= \emptyset;\\
\mathcal{I}(f(e_1,\dots,e_p)) &= f(\mathcal{I}(e_1),\dots,\mathcal{I}(e_p));\\
\mathcal{I}(\sim e) &= T(\mathcal{F}) \setminus \mathcal{I}(e);\\
\mathcal{I}(e \cup e') &= \mathcal{I}(e) \cup \mathcal{I}(e');\\
\mathcal{I}(e \cap e') &= \mathcal{I}(e) \cap \mathcal{I}(e').
\end{aligned}$$

We deduce an interpretation of set constraints in $\mathsf{Bool} = \{0,1\}$, the Boolean values. For a system of set constraints SC, all the interpretations $\mathcal{I}$ such that $\mathcal{I}(SC) = 1$ are called *solutions* of SC. In the remainder, we will consider systems of set constraints of n variables $X_1,\dots,X_n$. We will make no distinction between a *solution* $\mathcal{I}$ of a system of set constraints and a *n-tuple of tree languages* $(\mathcal{I}(X_1),\dots,\mathcal{I}(X_n))$. We denote by $\mathsf{SOL}(SC)$ the set of all solutions of a system of set constraints SC.

5.4.2 Set constraints and automata

Proposition 44. *Let SC be a system of set constraints (respectively of positive set constraints) of n variables $X_1,\dots,X_n$. There exists a deterministic (respectively deterministic and simple) generalized tree set automaton $\mathcal{A}$ over $\{0,1\}^n$ such that $\mathcal{L}(\mathcal{A})$ is the set of characteristic generalized tree sets of the n-tuples $(L_1,\dots,L_n)$ of solutions of SC.*

Proof. First we reduce the problem to a single set constraint. Let $SC = C_1 \wedge \dots \wedge C_k$ be a system of set constraints. A solution of SC satisfies all the constraints C_i. Let us suppose that, for every i, there exists a deterministic generalized tree set automaton $\mathcal{A}_i$ such that $\mathsf{SOL}(C_i) = \mathcal{L}(\mathcal{A})$. As all variables in $\{X_1,\dots,X_n\}$ do not necessarily occur in C_i, using Corollary 7, we can construct a deterministic generalized tree set automaton $\mathcal{A}_i^n$ over $\{0,1\}^n$ satisfying: $\mathcal{L}(\mathcal{A}_i^n)$ is the set of $(L_1,\dots,L_n)$ which corresponds to solutions of C_i when restricted to the variables of C_i. Using closure under intersection (Proposition 35), we can construct a deterministic generalized tree set automaton $\mathcal{A}$ over $\{0,1\}^n$ such that $\mathsf{SOL}(SC) = \mathcal{L}(\mathcal{A})$.

Therefore we prove the result for a set constraint SC of n variables $X_1,\dots,X_n$. Let $\mathcal{E}(exp)$ be the set of set variables and of set expression exp with a root symbol in $\mathcal{F}$ which occur in the set expression exp:

$$\mathcal{E}(exp) = \big\{exp' \in T_{\mathcal{F}'}(\mathcal{X}) \mid exp' \trianglelefteq exp \text{ and such that } \text{either } \mathcal{H}ead(exp') \in \mathcal{F} \text{ or } exp' \in \mathcal{X}\big\}.$$

If $SC \equiv exp_1 \subseteq exp_2$ or $SC \equiv exp_1 \not\subseteq exp_2$ then $\mathcal{E}(SC) = \mathcal{E}(exp_1) \cup \mathcal{E}(exp_2)$.

Let us consider a set constraint SC and let φ be a mapping φ from $\mathcal{E}(SC)$ into Bool. Such a mapping is easily extended first to any set expression occurring in SC and second to the set constraint SC. The symbols $\cup$, $\cap$, $\sim$, $\subseteq$ and $\not\subseteq$ are respectively interpreted as $\vee$, $\wedge$, $\neg$, $\Rightarrow$ and $\neg\Rightarrow$.

We now define the generalized tree set automaton $\mathcal{A} = (Q, \Delta, \Omega)$ over $E = \{0,1\}^n$.

- The set of states is Q is the set $\{\varphi \mid \varphi : \mathcal{E}(SC) \to \mathsf{Bool}\}$.
- The transition relation is defined as follows: $f(\varphi_1, \dots, \varphi_p) \xrightarrow{l} \varphi \in \Delta$ where $\varphi_1, \dots, \varphi_p \in Q$, $f \in \mathcal{F}_p$, $l = (l_1, \dots, l_n) \in \{0,1\}^n$, and $\varphi \in Q$ satisfies:

$$\forall i \in \{1, \dots, n\} \; \varphi(X_i) = l_i \tag{5.6}$$

$$\forall e \in \mathcal{E}(SC) \setminus \mathcal{X} \;\; (\varphi(e) = 1) \Leftrightarrow \left(\begin{array}{l} e = f(e_1, \dots, e_p) \\ \forall i \quad 1 \leq i \leq p \quad \varphi_i(e_i) = 1 \end{array} \right) \tag{5.7}$$

- The set of accepting sets of states Ω is defined depending on the case of a positive or a negative set constraint.
  - If SC is positive, $\Omega = \{\omega \in 2^Q \mid \forall \varphi \in \omega \; \varphi(SC) = 1\}$;
  - If SC is negative, $\Omega = \{\omega \in 2^Q \mid \exists \varphi \in \omega \; \varphi(SC) = 1\}$.

In the case of a positive set constraint, we can choose the state set $Q = \{\varphi \mid \varphi(SC) = 1\}$ and $\Omega = 2^Q$. Consequently, $\mathcal{A}$ is deterministic and simple.

The correctness of this construction is easy to prove and is left to the reader. □

5.4.3 Decidability results for set constraints

We now summarize results on set constraints. These results are immediate consequences of the results of Section 5.4.2.

Proposition 45.

1. *The satisfiability problem for systems of set constraints is decidable. There exists a regular solution, that is a tuple of regular tree languages, in any non-empty set of solutions.*
2. *Given two systems of set constraints SC and SC', it is decidable whether or not $\mathsf{SOL}(SC) \subseteq \mathsf{SOL}(SC')$.*
3. *Given a system SC of set constraints, it is decidable whether or not there is a unique solution in $\mathsf{SOL}(SC)$.*
4. *Given a system SC of set constraints over $(X_1, \dots, X_n)$ and $I \subseteq \{1, \dots, n\}$, it is decidable whether or not there is a solution $(L_1, \dots, L_n) \in \mathsf{SOL}(SC)$ such that all the L_i are finite for all i.*
5. *Given SC a system of set constraints over $(X_1, \dots, X_n)$ and a n-tuple $(L_1, \dots, L_n)$ of regular tree languages, it is decidable whether or not $(L_1, \dots, L_n) \in \mathsf{SOL}(SC)$.*

Proof. We use Proposition 44 to encode sets of solutions of systems of set constraints with generalized tree set automata and then, each point is deduced from Theorem 41, or Propositions 38, 43, 40, 41. □

Proposition 46. *Let SC be a system of positive set constraints, it is decidable whether or not there is a least solution in $\mathsf{SOL}(SC)$.*

Proof. Let SC be a system of positive set constraints. Let $\mathcal{A}$ be the deterministic, simple generalized tree set automaton over $\{0,1\}^n$ such that $\mathcal{L}(\mathcal{A}) = \mathsf{SOL}(SC)$ (see Proposition 44). We define a partial ordering $\preceq$ on $\mathcal{G}_{\{0,1\}^n}$ by :

$$\begin{array}{ll} \forall l, l' \in \{0,1\}^n & l \preceq l' \Leftrightarrow (\forall i \; l(i) \leq l'(i)) \\ \forall g, g' \in \mathcal{G}_{\{0,1\}^n} & g \preceq g' \Leftrightarrow (\forall t \in T(\mathcal{F}) \; g(t) \preceq g'(t)) \end{array}$$

The problem we want to deal with is to decide whether or not there exists a least generalized tree set w.r.t. $\preceq$ in $\mathcal{L}(\mathcal{A})$. To this aim, we first build a minimal solution if it exists, and second, we verify that this solution is unique.

Let ω be a subset of states such that $\mathsf{COND}(\omega)$ (see the sketch of proof page 150). Let $\mathcal{A}_\omega = (\omega, \Delta_\omega, 2^\omega)$ be the generalized tree set automaton $\mathcal{A}$ restricted to state set ω.

Now let ${\Delta_\omega}^{min}$ defined by: for each $(q_1, \ldots, q_p, f) \in \omega^p \times \mathcal{F}_p$, choose in the set Δ_ω one rule $(q_1, \ldots, q_p, f, l, q)$ such that l is minimal w.r.t. $\preceq$. Let ${\mathcal{A}_\omega}^{min} = (\omega, {\Delta_\omega}^{min}, 2^\omega)$. Consequently,

1. There exists only one run r_ω on a unique generalized tree set g_ω in ${\mathcal{A}_\omega}^{min}$ because for all $q_1, \ldots, q_p \in \omega$ and $f \in \mathcal{F}_p$ there is only one rule $(q_1, \ldots, q_p, f, l, q)$ in ${\Delta_\omega}^{min}$;

2. the run r_ω on g_ω is regular;

3. the generalized tree set g_ω is minimal w.r.t. $\preceq$ in $\mathcal{L}(\mathcal{A}_\omega)$.

Points 1 and 2 are straightforward. The third point follows from the fact that $\mathcal{A}$ is deterministic. Indeed, let us suppose that there exists a run r' on a generalized tree set g' such that $g' \prec g_\omega$. Therefore, $\forall t \; g'(t) \preceq g_\omega(t)$, and there exists (w.l.o.g.) a minimal term $u = f(u_1, \ldots, u_p)$ w.r.t. the subterm ordering such that $g'(u) \prec g_\omega(u)$. Since $\mathcal{A}$ is deterministic and $\forall v \lhd u \; g_\omega(v) = g'(v)$, we have $r_\omega(u_i) = r'(u_i)$. Hence, the rule $(r_\omega(u_1), \ldots, r_\omega(u_p), f, g_\omega(u), r_\omega(u))$ is not such that $g_\omega(u)$ is minimal in Δ_ω, which contradicts the hypothesis.

Consider the generalized tree sets g_ω for all subsets of states ω satisfying $\mathsf{COND}(\omega)$. It is decidable whether or not there is a minimal generalized tree set among them (each defines a n-tuple of regular tree languages and inclusion is decidable for regular tree languages). Either there exists a minimal generalized tree set g, then g is minimal in $\mathcal{L}(\mathcal{A})$ and it defines a n-tuple $(F_1, \ldots, F_n)$ of regular tree languages, or not and in this case there is no least generalized tree set g in $\mathcal{L}(\mathcal{A})$. If exists, there is a deterministic, simple generalized tree set automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}')$ is the set of characteristic generalized tree sets of all $(L_1, \ldots, L_n)$ satisfying $F_1 \subseteq L_1, \ldots, F_n \subseteq L_n$ (see Proposition 38). Let $\mathcal{A}''$ be the deterministic generalized tree set automaton such that $\mathcal{L}(\mathcal{A}'') = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$ (see Proposition 35). There exists a least generalized tree set w.r.t. $\preceq$ in $\mathcal{L}(\mathcal{A})$ if and only if the generalized tree set automata $\mathcal{A}$ and $\mathcal{A}''$ are equivalent. Since equivalence of generalized tree set automata is decidable (see Proposition 40) we get the result. □

5.5 Exercises

5.6 Bibliographical notes

We now survey decidability results for satisfiability of set constraints and some complexity issues.

Decision procedures for solving set constraints arise with [Rey69], and Mishra [Mis84]. The aim of these works was to obtain new tools for type inference and type checking [AM91, Hei92, HJ90b, JM79, Mis84, Rey69].

First consider systems of set constraints of the form:

$$X_1 = exp_1, \dots , X_n = exp_n \tag{5.8}$$

where the X_i are distinct variables and the exp_i are disjunctions of set expressions of the form $f(X_{i_1}, \dots , X_{i_p})$ with $f \in \mathcal{F}_p$. These systems of set constraints are essentially tree automata, therefore they have a unique solution and each X_i is interpreted as a regular tree language.

Suppose now that the exp_i are set expressions without complement symbols. Such systems are always satisfiable and have a least solution which is regular. For example, the system

$$\begin{aligned}&\mathsf{Nat} = s(\mathsf{Nat}) \cup 0\\ &X = X \cap \mathsf{Nat}\\ &\mathsf{List} = \mathsf{cons}(X, \mathsf{List}) \cup \mathsf{nil}\end{aligned}$$

has a least solution

$$\mathsf{Nat} = \{s^i(0) \mid i \geq 0\}\,, X = \emptyset\,, \mathsf{List} = \{\mathsf{nil}\}.$$

[HJ90a] investigate the class of definite set constraints which are of the form $exp \subseteq exp'$, where no complement symbol occurs and exp' contains no set operation. Definite set constraints have a least solution whenever they have a solution. The algorithm presented in [HJ90a] provides a specific set of transformation rules and, when there exists a solution, the result is a regular presentation of the least solution, in other words a system of the form (5.8).

Solving definite set constraints is EXPTIME-complete [CP97]. Many developments or improvements of Heinzte and Jaffar's method have been proposed and some are based on tree automata [DTT97].

The class of positive set constraints is the class of systems of set constraints of the form $exp \subseteq exp'$, where no projection symbol occur. In this case, when a solution exists, set constraints do not necessarily have a least solution. Several algorithms for solving systems in this class were proposed, [AW92] generalize the method of [HJ90a], [GTT93] give an automata-based algorithm, and [BGW93] use the decision procedure for the first order theory of monadic predicates. Results on the computational complexity of solving systems of set constraints are presented in a paper of [AKVW93]. The systems form a natural complexity hierarchy depending on the number of elements of $\mathcal{F}$ of each arity. The problem of existence of a solution of a system of positive set constraints is NEXPTIME-complete.

The class of positive and negative set constraints is the class of systems of set constraints of the form $exp \subseteq exp'$ or $exp \not\subseteq exp'$, where no projection symbol occur. In this case, when a solution exists, set constraints do not necessarily have, neither a minimal solution, nor a maximal solution. Let $\mathcal{F} = \{a, b()\}$. Consider the system $(b(X) \subseteq X) \wedge (X \not\subseteq \bot)$, this system has no minimal solution. Consider the system $(X \subseteq b(X) \cup a) \wedge (\top \not\subseteq X)$, this system has no maximal solution. The satisfiability problem in this class turned out to be much more difficult than the positive case. [AKW95] give a proof based on a reachability problem involving Diophantine inequalities. NEXPTIME-completeness was proved by [Ste94]. [CP94a] gives a proof based on the ideas of [BGW93].

The class of positive set constraints with projections is the class of systems of set constraints of the form $exp \subseteq exp'$ with projection symbols. Set constraints of the form $f_i^{-1}(X) \subseteq Y$ can easily be solved, but the case of set constraints of the form $X \subseteq f_i^{-1}(Y)$ is more intricate. The problem was proved decidable by [CP94b].

The expressive power of these classes of set constraints have been studied and have been proved to be different [Sey94]. In [CK96, Koz93], an axiomatization is proposed which enlightens the reader on relationships between many approaches on set constraints.

Furthermore, set constraints have been studied in a logical and topological point of view [Koz95, MGKW96]. This last paper combine set constraints with Tarskian set constraints, a more general framework for which many complexity results are proved or recalled. Tarskian set constraints involve variables, relation and function symbols interpreted relative to a first order structure.

Topological characterizations of classes of GTSA recognizable sets, have also been studied in [Tom94, Sey94]. Every set in $\mathcal{R}_{\mathrm{SGTS}}$ is a compact set and every set in $\mathcal{R}_{\mathrm{GTS}}$ is the intersection between a compact set and an open set. These remarks give also characterizations for the different classes of set constraints.

Chapter 6

Tree transducers

6.1 Introduction

Finite state transformations of words, also called a-transducers or rational transducers in the literature, model many kinds of processes, such as coffee machines or lexical translators. But these transformations are not powerful enough to model syntax directed transformations, and compiler theory is an important motivation to the study of finite state transformations of trees. Indeed, translation of natural or computing languages is directed by syntactical trees, and a translator from into HTML is a tree transducer. Unfortunately, from a theoretical point of view, tree transducers do not inherit nice properties of word transducers, and the classification is very intricate. So, in the present chapter we focus on some aspects. In Sections 6.2 and 6.3, toy examples introduce in an intuitive way different kinds of transducers. In Section 6.2, we summarize main results in the word case. Indeed, this book is mainly concerned with trees, but the word case is useful to understand the tree case and its difficulties. The bimorphism characterization is the ideal illustration of the link between the "machine" point of view and the "homomorphic" one. In Section 6.3, we motivate and illustrate bottom-up and top-down tree transducers, using compilation as leitmotiv. We precisely define and present the main classes of tree transducers and their properties in Section 6.4, where we observe that general classes are not closed under composition, mainly because of alternation of copying and nondeterministic processing. Nevertheless most useful classes, as those used in Section 6.3, have closure properties. In Section 6.5 we present the homomorphic point of view.

Most of the proofs are tedious and are omitted. This chapter is a very incomplete introduction to tree transducers. Tree transducers are extensively studied for themselves and for various applications. But as they are somewhat complicated objects, we focus here on the definitions and main general properties. It is usefull for every theoretical computer scientist to know main notions about tree transducers, because they are the main model of syntax directed manipulations, and that the heart of sofware manipulations and interfaces are syntax directed. Tree transducers are an essential frame to develop practical modular syntax directed algorithms, thought an effort of algorithmic engineering remains to do. Tree transducers theory can be fertilized by other area or can be usefull for other areas (example: Ground tree transducers for decidability of the first order theory of ground rewriting). We will be happy if after reading this chapter, the reader wants for further lectures, as monograph of Z. Fülöp and H. Vögler (december 1998 [FV98]).

6.2 The word case

6.2.1 Introduction to rational transducers

We suppose that the reader roughly knows popular notions of language theory: homomorphisms on words, finite automata, rational expressions, regular grammars. See for example the recent survey of A. Mateescu and A. Salomaa [MS96]. A rational transducer is a finite word automaton W with output. In a word automaton, a transition rule $f(q) \to q'(f)$ means "if W is in some state q, if it reads the input symbol f, then it enters state q' and moves its head one symbol to the right". For defining a rational transducer, it suffices to add an output, and a transition rule $f(q) \to q'(m)$ means "if the transducer is in some state q, if it reads the input symbol f, then it enters state q', writes the word m on the output tape, and moves its head one symbol to the right". Remark that with these notations, we identify a finite automaton with a rational transducer which writes what it reads. Note that m is not necessarily a symbol but can be a word, including the empty word. Furthermore, we assume that it is not necessary to read an input symbol, i.e. we accept transition rules of the form $\varepsilon(q) \to q'(m)$ (ε denotes the empty word).

Graph presentations of finite automata are popular and convenient. So it is for rational transducers. The rule $f(q) \to q'(m)$ will be drawn

$$\begin{tikzcd} \boxed{q} \arrow[r, "f/m"] & \boxed{q'} \end{tikzcd}$$

Example 51. **(Language L_1)** Let $\mathcal{F} = \{\langle, \rangle, ; , 0, 1, A, ..., Z\}$. In the following, we will consider the language L_1 defined on $\mathcal{F}$ by the regular grammar (the axiom is **program**):

program $\to \langle$ **instruct**
instruct $\to$ `LOAD` **register** | `STORE` **register** | `MULT` **register**
$\to$ | `ADD` **register**
register $\to$ 1**tailregister**
tailregister $\to$ 0**tailregister** | 1**tailregister** | ; **instruct** | $\rangle$

($a \to b|c$ is an abbreviation for the set of rules $\{a \to b, a \to c\}$)

L_1 is recognized by deterministic automaton A_1 of Figure 6.1. Semantic of L_1 is well known: `LOAD i` loads the content of register `i` in the accumulator; `STORE i` stores the content of the accumulator in register `i`; `ADD i` adds in the accumulator the content of the accumulator and the content of register `i`; `MULT i` multiplies in the accumulator the content of the accumulator and the content of register `i`.

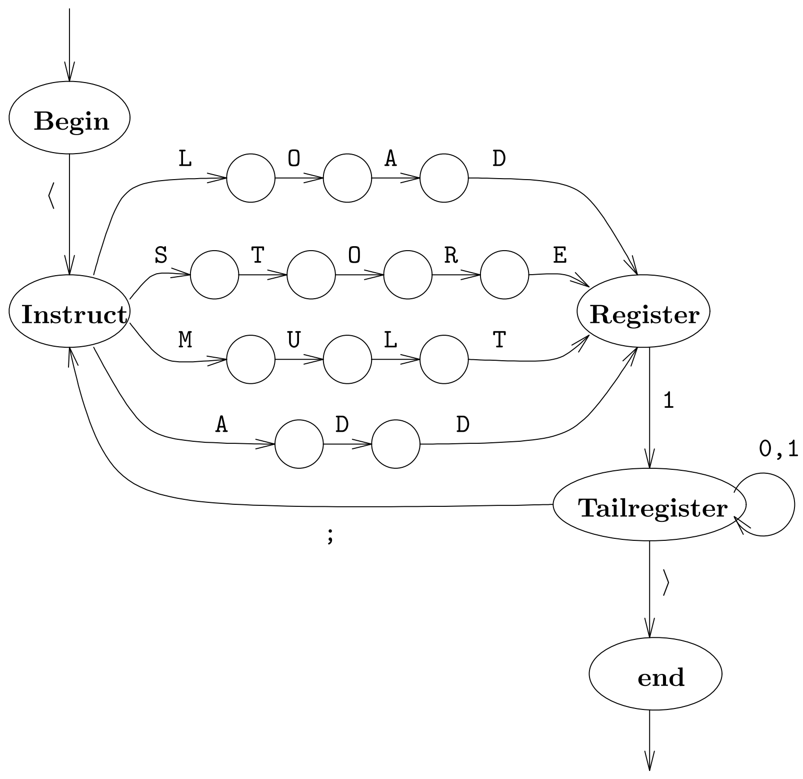


Figure 6.1: A recognizer of L_1

A ***rational transducer*** is a tuple $R = (Q, \mathcal{F}, \mathcal{F}', Q_i, Q_f, \Delta)$ where Q is a set of states, $\mathcal{F}$ and $\mathcal{F}'$ are finite nonempty sets of input letters and output letters, $Q_i, Q_f \subseteq Q$ are sets of initial and final states and Δ is a set of transduction rules of the following type:

$$f(q) \rightarrow q'(m),$$

where $f \in \mathcal{F} \cup \{\varepsilon\}$, $m \in \mathcal{F}'^*$, $q, q' \in Q$.

R is ***ε-free*** if there is no rule $f(q) \rightarrow q'(m)$ with $f = \varepsilon$ in Δ.

The ***move relation*** $\rightarrow_R$ is defined by: let $t, t' \in \mathcal{F}^*$, $u \in \mathcal{F}'^*$, $q, q' \in Q$, $f \in \mathcal{F}$, $m \in \mathcal{F}'^*$,

$$(tqft', u) \underset{R}{\rightarrow} (tfq't, um) \Leftrightarrow f(q) \rightarrow q'(m) \in \Delta,$$

and $\rightarrow_R^*$ is the reflexive and transitive closure of $\rightarrow_R$. A (partial) transduction of R on $tt't''$ is a sequence of move steps of the form $(tqt't'', u) \rightarrow_R^* (tt'q't'', uu')$. A transduction of R from $t \in \mathcal{F}^*$ into $u \in \mathcal{F}'^*$ is a transduction of the form $(qt, \varepsilon) \rightarrow_R^* (tq', u)$ with $q \in Q_i$ and $q' \in Q_f$.

The relation T_R induced by R can now be formally defined by:

$$T_R = \{(t, u) \mid (qt, \varepsilon) \underset{R}{\overset{*}{\rightarrow}} (tq', u) \text{ with } t \in \mathcal{F}^*, u \in \mathcal{F}'^*, q \in Q_i, q' \in Q_f\}.$$

A relation in $\mathcal{F}^* \times \mathcal{F}'^*$ is a rational transduction if and only if it is induced by some rational transducer. We also need the following definitions: let $t \in \mathcal{F}^*$, $T_R(t) = \{u \mid (t, u) \in T_R\}$. The translated of a language L is obviously the language defined by $T_R(L) = \{u \mid \exists t \in L, u \in T_R(t)\}$.

Example 52.

Ex. 52.1 Let us name French-L_1 the translation of L_1 in French (`LOAD` is translated into `CHARGER` and `STORE` into `STOCKER`). Transducer of Figure 6.2 realizes this translation. This example illustrates the use of rational transducers as lexical transducers.

Ex. 52.2 Let us consider the rational transducer *Diff* defined by $Q = \{q_i, q_s, q_l, q_d\}$, $\mathcal{F} = \mathcal{F}' = \{a, b\}$, $Q_i = \{q_i\}$, $Q_f = \{q_s, q_l, q_d\}$, and Δ is the set of rules:

type i $a(q_i) \rightarrow q_i(a)$, $b(q_i) \rightarrow q_i(b)$

type s $\varepsilon(q_i) \rightarrow q_s(a)$, $\varepsilon(q_i) \rightarrow q_s(b)$, $\varepsilon(q_s) \rightarrow q_s(a)$, $\varepsilon(q_s) \rightarrow q_s(b)$

type l $a(q_i) \rightarrow q_l(\varepsilon)$, $b(q_i) \rightarrow q_l(\varepsilon)$, $a(q_l) \rightarrow q_l(\varepsilon)$, $b(q_l) \rightarrow q_l(\varepsilon)$

type d $a(q_i) \rightarrow q_d(b)$, $b(q_i) \rightarrow q_d(a)$, $a(q_d) \rightarrow q_d(\varepsilon)$, $b(q_d) \rightarrow q_d(\varepsilon)$, $\varepsilon(q_d) \rightarrow q_d(a)$, $\varepsilon(q_d) \rightarrow q_d(b)$.

It is easy to prove that $T_{Diff} = \{(m, m') \mid m \neq m', m, m' \in \{a, b\}^*\}$.

We give without proofs some properties of rational transducers. For more details, see [Sal73] or [MS96] and Exercises 66, 67, 69 for 1, 4 and 5. The homomorphic approach presented in the next section can be used as an elegant way to prove 2 and 3 (Exercise 71).

Proposition 47 (Main properties of rational transducers).

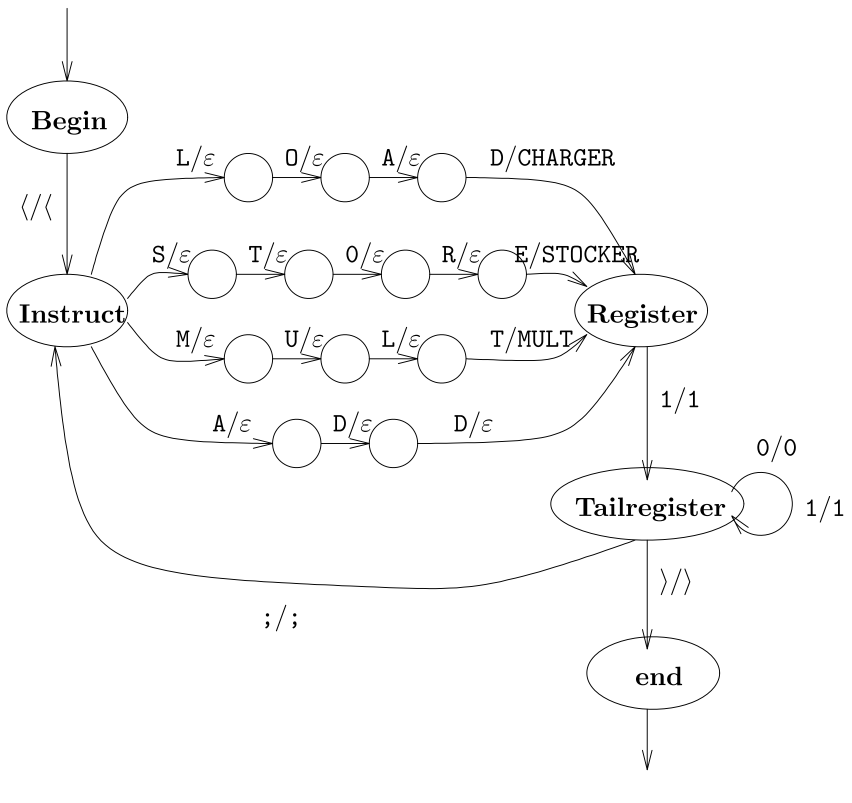


Figure 6.2: A rational transducer from L_1 into French-L_1.

1. *The class of rational transductions is closed under union but not closed under intersection.*
2. *The class of rational transductions is closed under composition.*
3. *Regular languages and context-free languages are closed under rational transduction.*
4. *Equivalence of rational transductions is undecidable.*
5. *Equivalence of deterministic rational transductions is decidable.*

6.2.2 The homomorphic approach

A ***bimorphism*** is defined as a triple $B = (\Phi, L, \Psi)$ where L is a recognizable language and Φ and Ψ are homomorphisms. The relation induced by B (also denoted by B) is defined by $B = \{(\Phi(t), \Psi(t)) \mid t \in L\}$. Bimorphism (Φ, L, Ψ) is ε-free if Φ is ε-free (an homomorphism is ε-free if the image of a letter is never reduced to ε). Two bimorphisms are equivalent if they induce the same relation.

We can state the following theorem, generally known as Nivat Theorem [Niv68] (see Exercises 70 and 71 for a sketch of proof).

Theorem 42 (Bimorphism theorem). *Given a rational transducer, an equivalent bimorphism can be constructed. Conversely, any bimorphism defines a rational transduction. Construction preserve ε-freeness.*

Example 53.

Ex. 53.1 The relation $\{(a(ba)^n, a^n) \mid n \in \mathbb{N}\} \cup \{((ab)^n, b^{3n}) \mid n \in \mathbb{N}\}$ is processed by transducer R and bimorphism B of Figure 6.3

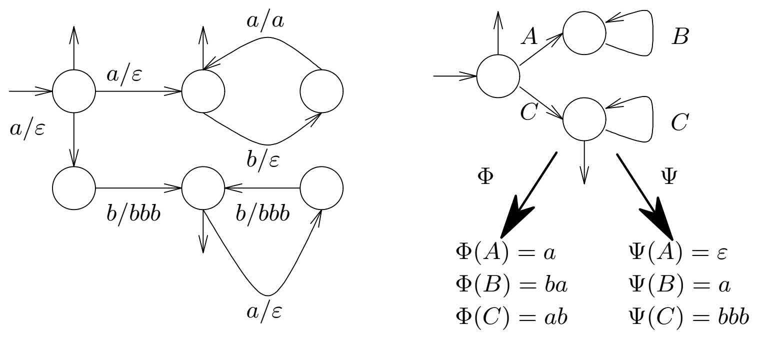


Figure 6.3: Transducer R and an equivalent bimorphism $B = \{(\Phi(t), \Psi(t)) \mid t \in AB^* + CC^*\}$.

Ex. 53.2 Automaton L of Figure 6.4 and morphisms Φ and Ψ bellow define a bimorphism equivalent to transducer of Figure 6.2

$\Phi(\beta) = \langle$	$\Phi(\lambda) = \texttt{LOAD}$	$\Phi(\sigma) = \texttt{STORE}$	$\Phi(\mu) = \texttt{MULT}$
$\Phi(\alpha) = \texttt{ADD}$	$\Phi(\rho) =;$	$\Phi(\omega) = 1$	$\Phi(\zeta) = 0$
$\Phi(\theta) =\rangle$			
$\Psi(\beta) = \langle$	$\Psi(\lambda) = \texttt{CHARGER}$	$\Psi(\sigma) = \texttt{STOCKER}$	$\Psi(\mu) = \texttt{MULT}$
$\Psi(\alpha) = \texttt{ADD}$	$\Psi(\rho) =;$	$\Psi(\omega) = 1$	$\Psi(\zeta) = 0$
$\Psi(\theta) =\rangle$			

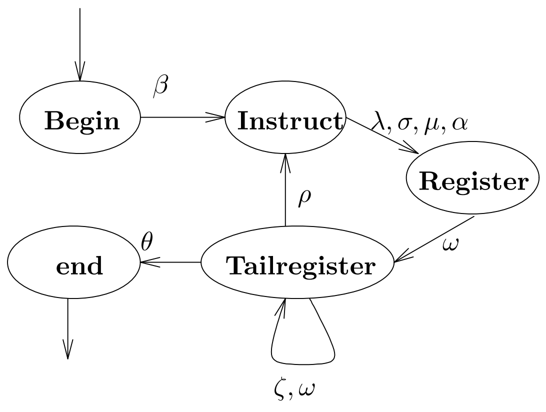


Figure 6.4: The control automaton L.

Nivat characterization of rational transducers makes intuitive sense. Automaton L can be seen as a control of the actions, morphism Ψ can be seen as output function and Φ^{-1} as an input function. Φ^{-1} analyses the input — it is a kind of part of lexical analyzer — and it generates symbolic names; regular grammatical structure on theses symbolic names is controlled by L. Examples 53.1 and 53.2 are an obvious illustration. L is the common structure to English and French versions, Φ generates the English version and Ψ generates the French one. This idea is the major idea of compilation, but compilation of computing languages or translation of natural languages are directed by syntax, that is to say by syntactical trees. This is the motivation of the rest of the chapter. But unfortunately, from a formal point of view, we will lose most of the best results of the word case. Power of non-linear tree transducers will explain in part this complication, but even in the linear case, there is a new phenomena in trees, the understanding of which can be introduced by the "problem of homomorphism inversion" that we describe in Exercise 72.

6.3 Introduction to tree transducers

Tree transducers and their generalizations model many syntax directed transformations (see exercises). We use here a toy example of compiler to illustrate how usual tree transducers can be considered as modules of compilers.

We consider a simple class of arithmetic expressions (with usual syntax) as source language. We suppose that this language is analyzed by a `LL1` parser. We consider two target languages: L_1 defined in Example 51 and an other language L_2. A transducer A translates syntactical trees in abstract trees (Figure 6.5). A second tree transducer R illustrates how tree transducers can be seen as part of compilers which compute attributes over abstract trees. It decorates abstract trees with numbers of registers (Figure 6.7). Thus R translates abstract trees into attributed abstract trees. After that, tree transducers T_1 and T_2 generate target programs in L_1 and L_2, respectively, starting from attributed abstract trees (Figures 6.7 and 6.8). This is an example of nonlinear transducer. Target programs are yields of generated trees. So composition of transducers model succession of compilation passes, and when a class of transducers is closed by composition (see section 6.4), we get universal constructions to reduce the number of compiler passes and to meta-optimize compilers.

We now define **the source language**. Let us consider the terminal alphabet $\{(,),+,\times,a,b,\dots,z\}$. First, the context-free word grammar G_1 is defined by rules (E is the axiom):

$$\begin{array}{rcl} E & \rightarrow & M \mid M+E \\ M & \rightarrow & F \mid F\times M \\ F & \rightarrow & I \mid (E) \\ I & \rightarrow & a \mid b \mid \cdots \mid z \end{array}$$

Another context-free word grammar G_2 is defined by (E is the axiom):

$$\begin{array}{rcl} E & \rightarrow & ME' \\ E' & \rightarrow & +E \mid \varepsilon \\ M & \rightarrow & FM' \\ M' & \rightarrow & \times M \mid \varepsilon \\ F & \rightarrow & I \mid (E) \\ I & \rightarrow & a \mid b \mid \cdots \mid z \end{array}$$

Let E be the axiom of G_1 and G_2. The semantic of these two grammars is obvious. It is easy to prove that they are equivalent, i.e. they define the same source language. On the one hand, G_1 is more natural, on the other hand G_2 could be preferred for syntactical analysis reason, because G_2 is LL1 and G_1 is not LL. We consider **syntactical trees** as derivation trees for the tree grammar G_2. Let us consider word $u = (a+b)\times c$. u of the source language. We define the abstract tree associated with u as the tree $\times(+(a,b),c)$ defined over $\mathcal{F} = \{+(,),\times(,),a,b,c\}$. **Abstract trees** are ground terms over $\mathcal{F}$. Evaluate expressions or compute attributes over abstract trees than over syntactical trees. The following transformation associates with a syntactical tree t its corresponding abstract tree $A(t)$.

$$\begin{array}{rr} I(x) \rightarrow x & F(x) \rightarrow x \\ M(x, M'(\varepsilon)) \rightarrow x & E(x, E'(\varepsilon)) \rightarrow x \\ M(x, M'(\times, y)) \rightarrow \times(x,y) & E(x, E'(+,y)) \rightarrow +(x,y) \\ F((,x,)) \rightarrow x & \end{array}$$

We have not precisely defined the use of the arrow $\rightarrow$, but it is intuitive. Likewise we introduce examples before definitions of different kinds of tree transducers (section 6.4 supplies a formal frame).

To illustrate nondeterminism, let us introduce two new transducers A and A'. Some brackets are optional in the source language, hence A' is nondeterministic. Note that A works from frontier to root and A' works fromm root to frontier.

A : an example of bottom-up tree transducer

The following linear deterministic bottom-up tree transducer A carries out transformation of derivation trees for G_2 into the corresponding abstract trees. Empty word ε is identified as a constant symbol in syntactical trees. States of A are q, q_ε, q_I, q_F, $q_{M'\varepsilon}$, $q_{E'\varepsilon}$, q_E, $q_\times$, $q_{M'\times}$, q_+, $q_{E'+}$, $q_($, and $q_)$. Final state is q_E. The set of transduction rules is:

$$
\begin{array}{rlrl}
a & \rightarrow q(a) & b & \rightarrow q(b) \\
c & \rightarrow q(c) & \varepsilon & \rightarrow q_\varepsilon(\varepsilon) \\
) & \rightarrow q_)()) & (& \rightarrow q_((() \\
+ & \rightarrow q_+(+) & \times & \rightarrow q_\times(\times) \\
I(q(x)) & \rightarrow q_I(x) & F(q_I(x)) & \rightarrow q_F(x) \\
M'(q_\varepsilon(x)) & \rightarrow q_{M'\varepsilon}(x) & E'(q_\varepsilon(x)) & \rightarrow q_{E'\varepsilon}(x) \\
M(q_F(x), q_{M'\varepsilon(y)}) & \rightarrow q_M(x) & E(q_M(x), q_{E'\varepsilon}(y)) & \rightarrow q_E(x) \\
M'(q_\times(x), q_M(y)) & \rightarrow q_{M'\times}(y) & M(q_F(x), q_{M'\times}(y)) & \rightarrow q_M(\times(x,y)) \\
E'(q_+(x), q_E(y)) & \rightarrow q_{E'+}(y) & E(q_M(x), q_{E'+}(y)) & \rightarrow q_E(+(x,y)) \\
F(q_((x), q_E(y), q_)(z)) & \rightarrow q_F(y) & &
\end{array}
$$

The notion of (successful) run is an intuitive generalization of the notion of run for finite tree automata. The reader should note that FTAs can be considered as a special case of bottom-up tree transducers whose output is equal to the input. We give in Figure 6.5 an example of run of A which translates derivation tree t which yields $(a + b) \times c$ for context-free grammar G_2 into the corresponding abstract tree $\times(+(a, b), c)$.

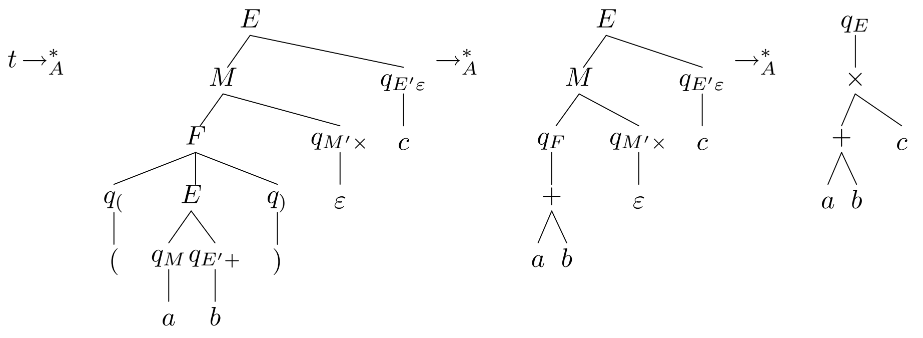


Figure 6.5: Example of run of A

A' : an example of top-down tree transducer

The inverse transformation A^{-1}, which computes the set of derivation trees of G_2 associated with an abstract tree, is computed by a nondeterministic top-down tree transducer A'. The states of A' are q_E, q_F, q_M. The initial state is q_E. The set of transduction rules is:

$$
\begin{array}{rlrl}
q_E(x) & \rightarrow E(q_M(x), E'(\varepsilon)) & q_E(+(x,y)) & \rightarrow E(q_M(x), E'(+, q_E(y))) \\
q_M(x) & \rightarrow M(q_F(x), M'(\varepsilon)) & q_M(\times(x,y)) & \rightarrow M(q_F(x), M'(\times, q_M(y))) \\
q_F(x) & \rightarrow F((, q_E(x),)) & q_F(a) & \rightarrow F(I(a)) \\
q_F(b) & \rightarrow F(I(b)) & q_F(c) & \rightarrow F(I(c))
\end{array}
$$

Transducer A' is nondeterministic because there are ε-rules like $q_E(x) \rightarrow E(q_M(x), E'(\varepsilon))$. We give in Figure 6.6 an example of run of A' which transforms abstract tree $+(a, \times(b,c))$ into a syntactical tree t' of the word $a + b \times c$.

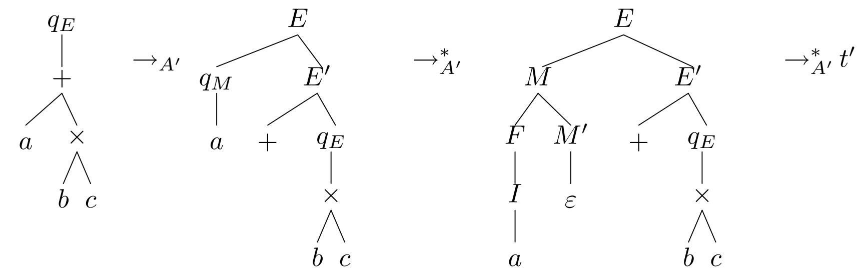


Figure 6.6: Example of run of A'

Compilation

The compiler now transforms abstract trees into programs for some target languages. We consider two target languages. The first one is L_1 of Example 51. To simplify, we omit ";", because they are not necessary — we introduced semicolons in Section 6.2 to avoid ε-rules, but this is a technical detail, because word (and tree) automata with ε-rules are equivalent to usual ones. The second target language is an other very simple language L_2, namely sequences of two instructions $+(i,j,k)$ (put the sum of contents of registers i and j in the register k) and $\times(i,j,k)$. In a first pass, we attribute to each node of the abstract tree the minimal number of registers necessary to compute the corresponding subexpression in the target language. The second pass generates target programs.

First pass: computation of register numbers by **a deterministic linear bottom-up transducer** R.

States of a tree automaton can be considered as values of (finitely valued) attributes, but formalism of tree automata does not allow decorating nodes of trees with the corresponding values. On the other hand, this decoration is easy with a transducer. Computation of finitely valued inherited (respectively synthesized) attributes is modeled by top-down (respectively bottom-up) tree transducers. Here, we use a bottom-up tree transducer R. States of R are $q_0, \ldots, q_n$. All states are final states. The set of rules is:

$$\begin{array}{ll}
a \rightarrow q_0(a) & b \rightarrow q_0(b) \\
c \rightarrow q_0(c) & \\
+(q_i(x), q_i(y)) \rightarrow q_{i+1}(\underset{i+1}{+}(x,y)) & \times(q_i(x), q_i(y)) \rightarrow q_{i+1}(\underset{i}{+}1 \times (x,y)) \\
\text{if } i > j & \\
+(q_i(x), q_j(y)) \rightarrow q_i(\underset{i}{+}(x,y) & \times(q_i(x), q_j(y)) \rightarrow q_i(\underset{i}{\times}(x,y)) \\
\text{if } i < j \text{, we permute the order of subtrees} & \\
+(q_i(x), q_j(y)) \rightarrow q_j(\underset{j}{+}(y,x)) & \times(q_i(x), q_j(y)) \rightarrow q_j(\underset{j}{\times}(y,x))
\end{array}$$

A run $t \rightarrow_R^* q_i(u)$ means that i registers are necessary to evaluate t. Root of t is then relabelled in u by symbol $\underset{i}{+}$ or $\underset{i}{\times}$.

Second pass: generation of target programs in L_1 or L_2, by **top-down deterministic transducers** T_1 **and** T_2. T_1 contains only one state q. Set of rules of T_1 is:

$$\begin{aligned}
q(\underset{i}{+}(x,y)) &\rightarrow \diamond(q(x), \mathtt{STORE}i, q(y), \mathtt{ADD}i, \mathtt{STORE}i) \\
q(\underset{i}{\times}(x,y)) &\rightarrow \diamond(q(x), \mathtt{STORE}i, q(y), \mathtt{MULT}i, \mathtt{STORE}i) \\
q(a) &\rightarrow \diamond(\mathtt{LOAD}, a) \\
q(b) &\rightarrow \diamond(\mathtt{LOAD}, b) \\
q(c) &\rightarrow \diamond(\mathtt{LOAD}, c)
\end{aligned}$$

where $\diamond(,,,,)$ and $\diamond(,)$ are new symbols.

State set of T_2 is $\{q, q'\}$ where q' is the initial state. Set of rules of T_2 is:

$$\begin{array}{rlrl}
q(\underset{i}{+}(x,y)) & \rightarrow \#(q(x), q(y), +, (, q'(x), q'(y), i,)) & q'(\underset{i}{+}(x,y)) & \rightarrow i \\
q(\underset{i}{\times}(x,y)) & \rightarrow \#(q(x), q(y), \times, (, q'(x), q'(y), i,)) & q'(\underset{i}{\times}(x,y)) & \rightarrow i \\
q(a) & \rightarrow \varepsilon & q'(a) & \rightarrow a \\
q(b) & \rightarrow \varepsilon & q'(b) & \rightarrow b \\
q(c) & \rightarrow \varepsilon & q'(c) & \rightarrow c
\end{array}$$

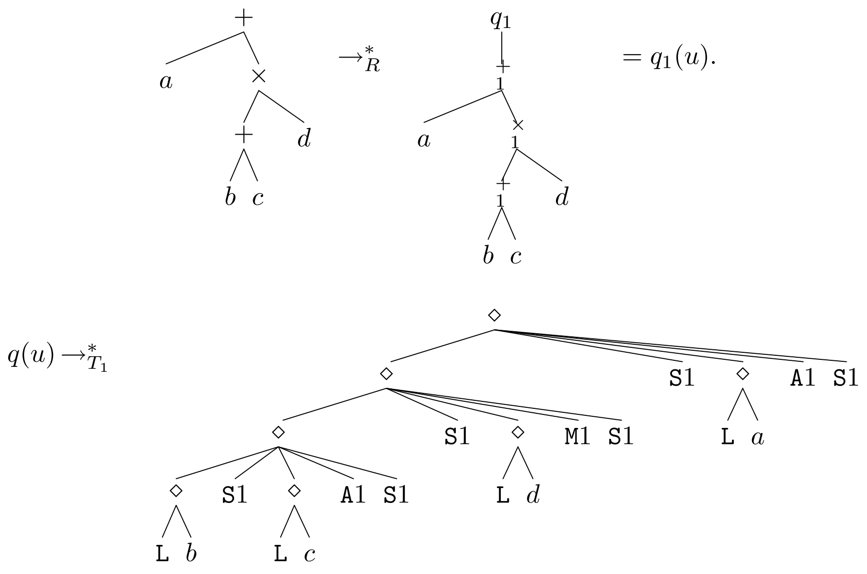
where # is a new symbol of arity 8.

The reader should note that target programs are words formed with leaves of trees, i.e. yields of trees. Examples of transductions computed by T_1 and T_2 are given in Figures 6.7 and 6.8. The reader should also note that T_1 is an homomorphism. Indeed, an homomorphism can be considered as a particular case of deterministic transducer, namely a transducer with only one state (we can consider it as bottom-up as well as top-down). The reader should also note that T_2 is deterministic but not linear.

6.4 Properties of tree transducers

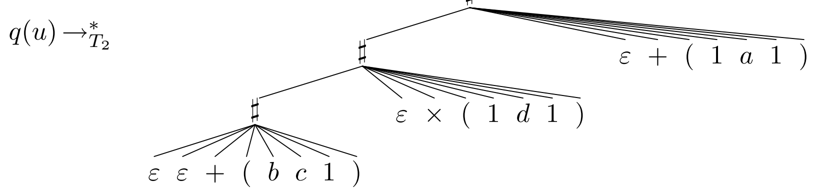
6.4.1 Bottom-up tree transducers

We now give formal definitions. In this section, we consider academic examples, without intuitive semantic, to illustrate phenomena and properties. Tree transducers are both generalization of word transducers and tree automata. We first



where `L` stands for `LOAD`, `S` stands for `STORE`, `A` stands for `ADD`, `M` stands for `MULT`. The corresponding program is the yield of this tree:
`LOAD`b `STORE`1 `LOAD`c `ADD`1 `STORE`1 `STORE`1 `LOAD`d `MULT`1 `STORE`1 `STORE`1 `LOAD`a `ADD`1 `STORE`1

Figure 6.7: Decoration with synthesized attributes of an abstract tree, and translation into a target program of L_1.



$q(u) \rightarrow_{T_2}^*$
$\sharp$
ε $+$ (1 a 1)
ε $\times$ (1 d 1)
ε ε $+$ (b c 1)

The corresponding program is the yield of this tree: $+(bc1) \times (1d1) + (1a1)$

Figure 6.8: Translation of an abstract tree into a target program of L_2

consider bottom-up tree transducers. A transition rule of a NFTA is of the type $f(q_1(x_1), \ldots, q_n(x_n)) \rightarrow q(f(x_1, \ldots, x_n))$. Here we extend the definition (as we did in the word case), accepting to change symbol f into any term.

A ***bottom-up Tree Transducer*** (NUTT) is a tuple $U = (Q, \mathcal{F}, \mathcal{F}', Q_f, \Delta)$ where Q is a set of (unary) states, $\mathcal{F}$ and $\mathcal{F}'$ are finite nonempty sets of input symbols and output symbols, $Q_f \subseteq Q$ is a set of final states and Δ is a set of transduction rules of the following two types:

$$f(q_1(x_1), \ldots, q_n(x_n)) \rightarrow q(u) \ ,$$

where $f \in \mathcal{F}_n$, $u \in T(\mathcal{F}', \mathcal{X}_n)$, $q, q_1, \ldots, q_n \in Q$, or

$$q(x_1) \rightarrow q'(u) \quad (\varepsilon\text{-rule}),$$

where $u \in T(\mathcal{F}', \mathcal{X}_1)$, $q, q' \in Q$.

As for NFTA, there is no initial state, because when a symbol is a leave a (i.e. a constant symbol), transduction rules are of the form $a \rightarrow q(u)$, where u is a ground term. These rules can be considered as "initial rules". Let $t, t' \in T(\mathcal{F} \cup \mathcal{F}' \cup Q)$. The move relation $\rightarrow_U$ is defined by:

$$t \underset{U}{\rightarrow} t' \Leftrightarrow \begin{cases} \exists f(q_1(x_1), \ldots, q_n(x_n)) \rightarrow q(u) \in \Delta \\ \exists C \in \mathcal{C}(\mathcal{F} \cup \mathcal{F}' \cup Q) \\ \exists u_1, \ldots, u_n \in T(\mathcal{F}') \\ t = C[f(q_1(u_1), \ldots, q_n(u_n))] \\ t' = C[q(u\{x_1 \leftarrow u_1, \ldots, x_n \leftarrow u_n\})] \end{cases}$$

This definition includes the case of ε-rule as a particular case. The reflexive and transitive closure of $\rightarrow_U$ is $\rightarrow_U^*$. A transduction of U from a ground term $t \in T(\mathcal{F})$ to a ground term $t' \in T(\mathcal{F}')$ is a sequence of move steps of the form $t \rightarrow_U^* q(t')$, such that q is a final state. The relation induced by U is the relation (also denoted by U) defined by:

$$U = \{(t, t') \mid t \overset{*}{\underset{U}{\rightarrow}} q(t'), t \in T(\mathcal{F}), t' \in T(\mathcal{F}'), q \in Q_f\}.$$

The domain of U is the set $\{t \in T(\mathcal{F}) \mid (t, t') \in U\}$. The image by U of a set of ground terms L is the set $U(L) = \{t' \in T(\mathcal{F}') \mid \exists t \in L, (t, t') \in U\}$.

A transducer is ***ε-free*** if it contains no ε-rule. It is ***linear*** if all transition rules are linear (no variable occurs twice in the right-hand side). It is ***non-erasing*** if, for every rule, at least one symbol of $\mathcal{F}'$ occurs in the right-hand side. It is said to be ***complete*** (or non-deleting) if, for every rule $f(q_1(x_1), \ldots, q_n(x_n)) \rightarrow q(u)$, for every $x_i (1 \leq i \leq n)$, x_i occurs at least once in u. It is ***deterministic*** (DUTT) if it is ε-free and there is no two rules with the same left-hand side.

Example 54.

Ex. 54.1 Tree transducer A defined in Section 6.3 is a linear DUTT. Tree transducer R in Section 6.3 is a linear and complete DUTT.

Ex. 54.2 States of U_1 are q, q'; $\mathcal{F} = \{f(), a\}$; $\mathcal{F}' = \{g(,), f(), f'(), a\}$; q' is the final state; the set of transduction rules is:

$$\begin{array}{c} a \rightarrow q(a) \\ f(q(x)) \rightarrow q(f(x)) \mid q(f'(x)) \mid q'(g(x, x)) \end{array}$$

U_1 is a complete, non linear NUTT. We now give the transductions of the ground term $f(f(f(a)))$. For the sake of simplicity, $fffa$ stands for $f(f(f(a)))$. We have:

$$U_1(\{fffa\}) = \{g(ffa, ffa), g(ff'a, ff'a), g(f'fa, f'fa), g(f'f'a, f'f'a)\}.$$

U_1 illustrates an ability of NUTT, that we describe following Gécseg and Steinby.

B1- "Nprocess and copy" A NUTT can first process an input subtree nondeterministically and then make copies of the resulting output tree.

Ex. 54.3 States of U_2 are q, q'; $\mathcal{F} = \mathcal{F}' = \{f(), f'(), a\}$; q is the final state; the set of transduction rules is defined by:

$$\begin{aligned} a &\to q(a) \\ f(q(x)) &\to q'(a) \\ f'(q'(x)) &\to q(a) \end{aligned}$$

U_2 is a non complete DUTT. The tree transformation induced by U_2 is

$$\left\{(t, a) \mid \begin{array}{l} t \text{ is accepted by the DFTA of final state } q \text{ and rules} \\ a \to q(a), f(q(x)) \to q'(f(x)), f'(q'(x)) \to q(f'(x)) \end{array} \right\}.$$

B2- "check and delete" A NUTT can first check regular constraints on input subterms and delete these subterms afterwards.

Bottom-up tree transducers translate the input trees from leaves to root, so bottom-up tree transducers are also called frontier-to-root transducers. Top-down tree transducers work in opposite direction.

6.4.2 Top-down tree transducers

A ***top-down Tree Transducer*** (NDTT) is a tuple $D = (Q, \mathcal{F}, \mathcal{F}', Q_i, \Delta)$ where Q is a set of (unary) states, $\mathcal{F}$ and $\mathcal{F}'$ are finite nonempty sets of input symbols and output symbols, $Q_i \subseteq Q$ is a set of initial states and Δ is a set of transduction rules of the following two types:

$$q(f(x_1, \dots, x_n)) \to u[q_1(x_{i_1}), \dots, q_p(x_{i_p})] ,$$

where $f \in \mathcal{F}_n$, $u \in \mathcal{C}^p(\mathcal{F}')$, q, $q_1, \dots, q_p \in Q$, , $x_{i_1}, \dots, x_{i_p} \in \mathcal{X}_n$, or

$$q(x) \to u[q_1(x), \dots, q_p(x)] \quad (\varepsilon\text{-rule}),$$

where $u \in \mathcal{C}^p(\mathcal{F}')$, q, $q_1, \dots, q_p \in Q$, $x \in \mathcal{X}$.

As for top-down NFTA, there is no final state, because when a symbol is a leave a (i.e. a constant symbol), transduction rules are of the form $q(a) \to u$, where u is a ground term. These rules can be considered as "final rules". Let $t, t' \in T(\mathcal{F} \cup \mathcal{F}' \cup Q)$. The move relation $\to_D$ is defined by:

$$t \underset{D}{\rightarrow} t' \Leftrightarrow \begin{cases} \exists q(f(x_1,\ldots,x_n)) \rightarrow u[q_1(x_{i_1}),\ldots,q_p(x_{i_p})] \in \Delta \\ \exists C \in \mathcal{C}(\mathcal{F} \cup \mathcal{F}' \cup Q) \\ \exists u_1,\ldots,u_n \in T(\mathcal{F}) \\ t = C[q(f(u_1,\ldots,u_n))] \\ t' = C[u[q_1(v_1),\ldots,q_p(v_p)]] \text{ where } v_j = u_k \text{ if } x_{i_j} = x_k \end{cases}$$

This definition includes the case of ε-rule as a particular case. $\rightarrow_D^*$ is the reflexive and transitive closure of $\rightarrow_D$. A transduction of D from a ground term $t \in T(\mathcal{F})$ to a ground term $t' \in T(\mathcal{F}')$ is a sequence of move steps of the form $q(t) \rightarrow_D^* t'$, where q is an initial state. The transformation induced by D is the relation (also denoted by D) defined by:

$$D = \{(t,t') \mid q(t) \overset{*}{\underset{D}{\rightarrow}} t', t \in T(\mathcal{F}), t' \in T(\mathcal{F}'), q \in Q_i\}.$$

The domain of D is the set $\{t \in T(\mathcal{F}) \mid (t,t') \in D\}$. The image of a set of ground terms L by D is the set $D(L) = \{t' \in T(\mathcal{F}') \mid \exists t \in L, (t,t') \in D\}$. ε-free, linear, non-erasing, complete (or non-deleting), deterministic top-down tree transducers are defined as in the bottom-up case.

Example 55.

Ex. 55.1 Tree transducers A', T_1, T_2 defined in Section 6.3 are examples of NDTT.

Ex. 55.2 Let us now define a non-deterministic and non linear NDTT D_1. States of D_1 are q, q'. The set of input symbols is $\mathcal{F} = \{f(), a\}$. The set of output symbols is $\mathcal{F}' = \{g(,), f(), f'(), a\}$. The initial state is q. The set of transduction rules is:

$$\begin{aligned} q(f(x)) &\rightarrow g(q'(x), q'(x)) && \text{(copying rule)} \\ q'(f(x)) &\rightarrow f(q'(x)) \mid f'(q'(x)) && \text{(non deterministic relabeling)} \\ q'(a) &\rightarrow a \end{aligned}$$

D_1 transduces $f(f(f(a)))$ (or briefly $fffa$) into the set of 16 trees:

$\{g(ffa, ffa), g(ffa, ff'a), g(ffa, f'fa), \ldots, g(f'f'a, f'fa), g(f'f'a, f'f'a)\}$.

D_1 illustrates a new property.

D- "copy and Nprocess" A NDTT can first make copies of an input subtree and then process different copies independently and nondeterministically .

6.4.3 Structural properties

In this section, we use tree transducers U_1, U_2 and D_1 of the previous section in order to point out differences between top-down and bottom-up tree transducers.

Theorem 43 (Comparison Theorem).

1. *There is no top-down tree transducer equivalent to U_1 or to U_2.*
2. *There is no bottom-up tree transducer equivalent to D_1.*
3. *Any linear top-down tree transducer is equivalent to a linear bottom-up tree transducer. In the linear complete case, classes of bottom-up and top-down tree transducers are equal.*

It is not hard to verify that neither NUTT nor NDTT are closed under composition. Therefore, comparison of D-property "copy and Nprocess" and U-property "Nprocess and copy" suggests an important question:

> *does alternation of copying and non-determinism induces an infinite hierarchy of transformations?*

The answer is affirmative [Eng78, Eng82], but it was a relatively long-standing open problem. The fact that top-down transducers copy before non-deterministic processes, and bottom-up transducers copy after non-deterministic processes (see Exercise 76) suggests too that we get by composition two intricate infinite hierarchies of transformation. The following theorem summarizes results.

Theorem 44 (Hierarchy theorem). *By composition of NUTT, we get an infinite hierarchy of transformations. Any composition of n NUTT can be processed by composition of $n+1$ NDTT, and conversely (i.e. any composition of n NDTT can be processed by composition of $n+1$ NUTT).*

Transducer A' of Section 6.3 shows that it can be useful to consider ε-rules, but usual definitions of tree transducers in literature exclude this case of non determinism. This does not matter, because it is easy to check that all important results of closure or non-closure hold simultaneously for general classes and ε-free classes. Deleting is also a minor phenomenon. Indeed, it gives rise to the "check and delete" property, which is specific to bottom-up transducers, but it does not matter for hierarchy theorem, which remains true if we consider complete transducers.

Section 6.3 suggests that for practical use, non-determinism and non-linearity are rare. Therefore, it is important to note than if we suppose linearity or determinism, hierarchy of Theorem 45 collapses. Following results supply algorithms to compose or simplify transducers.

Theorem 45 (Composition Theorem).

1. *The class of linear bottom-up transductions is closed under composition.*
2. *The class of deterministic bottom-up transductions is closed under composition.*
3. *The class of linear top-down transductions is included in the class of linear bottom-up transductions. These classes are equivalent in the complete case.*

4. *Any composition of deterministic top-down transductions is equivalent to a deterministic complete top-down transduction composed with a linear homomorphism.*

The reader should note that bottom-up determinism and top-down determinism are incomparable (see Exercise 73).

Recognizable tree languages play a crucial role because derivation trees of context-free word grammars are recognizable. Fortunately, we get:

Theorem 46 (Recognizability Theorem). *The domain of a tree transducer is a recognizable tree language. The image of a recognizable tree language by a linear tree transducer is recognizable.*

6.4.4 Complexity properties

We present now some decidability and complexity results. As for structural properties, the situation is more complicated than in the word case, especially for top-down tree transducers. Most of problems are untractable in the worst case, but empirically "not so much complex" in real cases, though there is a lake of "algorithmic engineering" to get performant algorithms. As in the word case, emptiness is decidable, and equivalence in undecidable in the general case but is decidable in the k-valued case (a transducer is k-valued if there is no tree which is transduced in more than k different terms; so a deterministic transducer is a particular case of 1-valued transducer).

Theorem 47 (Recidability and complexity). *Emptiness of tree transductions is decidable. Equivalence of k-valued tree transducers is decidable.*

Emptiness for bottom-up transducers is essentially the same as emptiness for tree automata and therefore PTIME complete. Emptiness for top-down automata, however, is essentially the same as emptiness for alternating topdown tree automata, giving DEXPTIME completeness for emptiness. The complexity PTIME for testing single-valuedness in the bottom-up case is contained in Seidl [Sei92]. Ramsey theory gives combinatorial properties onto which equivalence tests for k-valued tree transducers [Sei94a].

Theorem 48 (Equivalence Theorem). *Equivalence of deterministic tree transducers is decidable.*

6.5 Homomorphisms and tree transducers

Exercise 75 illustrates how decomposition of transducers using homomorphisms can help to get composition results, but we are far from the nice bimorphism theorem of the word case, and in the tree case, there is no illuminating theorem, but many complicated partial statements. Seminal paper of Engelfriet [Eng75] contains a lot of decomposition and composition theorems. Here, we only present the most significant results.

A delabeling is a linear, complete, and symbol-to-symbol tree homomorphism (see Section 1.4). This very special kind of homomorphism changes only the label of the input letter and possibly order of subtrees. Definition of tree bimorphisms is not necessary, it is the same as in the word case. We get the following characterization theorem. We say that a bimorphism is linear, (respectively complete, etc) if the two morphisms are linear, (respectively complete, etc).

Theorem 49. *The class of bottom-up tree transductions is equivalent to the class of bimorphisms (Φ, L, Ψ) where Φ is a delabeling.*

Relation defined by (Φ, L, Ψ) is computed by a transduction which is linear (respectively complete, ε-free) if Ψ is linear (respectively complete, ε-free).

Remark that Nivat Theorem illuminates the symmetry of word transductions: the inverse relation of a rational transduction is a rational transduction. In the tree case, non-linearity obviously breaks this symmetry, because a tree transducer can copy an input tree and process several copies, but it can never check equality of subtrees of an input tree. If we want to consider symmetric relations, we have two main situations. In the non-linear case, it is easy to prove that composition of two bimorphisms simulates a Turing machine. In the linear and the linear complete cases, we get the following results.

Theorem 50 (Tree Bimorphisms). .

1. *The class LCFB of linear complete ε-free tree bimorphisms satisfies $\mathsf{LCFB} \subset \mathsf{LCFB}^2 = \mathsf{LCFB}^3$.*

2. *The class LB of linear tree bimorphisms satisfies $\mathsf{LB} \subset \mathsf{LB}^2 \subset \mathsf{LB}^3 \subset \mathsf{LB}^4 = \mathsf{LB}^5$.*

Proof of $\mathsf{LCFB}^2 = \mathsf{LCFB}^3$ requires many refinements and we omit it.

To prove $\mathsf{LCFB} \subset \mathsf{LCFB}^2$ we use twice the same homomorphism $\Phi(a) = a, \Phi(f(x)) = f(x), \Phi(g(x,y)) = g(x,y)), \Phi(h(x,y,z)) = g(x, g(y,z))$.

For any subterms $(t_1, \dots, t_{2p+2})$, let

$$t = h(t_1, t_2, h(t_3, t_4, h(t_{2i+1}, t_{2i+2}, \dots, h(t_{2p-1}, t_{2p}, g(t_{2p+1}, t_{2p+2}) \dots)))$$

and

$$t' = g(t_1, h(t_2, t_3, h(t_4, \dots, h(t_{2i}, t_{2i+1}, h(t_{2i+2}, t_{2i+3}, \dots, h(t_{2p}, t_{2p+1}, t_{2p+2}) \dots))).$$

We get $t' \in (\Phi \circ \Phi^{-1})(t)$. Assume that $\Phi \circ \Phi^{-1}$ can be processed by some $\Psi^{-1} \circ \Psi'$. Consider for simplicity subterms t_i of kind $f^{ni}(a)$. Roughly, if lengths of t_i are different enough, Ψ and Ψ' must be supposed linear complete. Suppose that for some u we have $\Psi(u) = t$ and $\Psi'(u) = t'$, then for any context u' of u, $\Psi(u')$ is a context of t with an odd number of variables, and $\Psi'(u')$ is a context of t' with an even number of variables. That is impossible because homomorphisms are linear complete.

Point 2 is a refinement of point 1 (see Exercise 80).

This example shows a stronger fact: the relation cannot be processed by any bimorphism, even non-linear, nor by any bottom-up transducer A direct characterization of these transformations is given in [AD82] by a special class of top-down tree transducers, which are not linear but are "globally" linear, and which are used to prove $\mathsf{LCFB}^2 = \mathsf{LCFB}^3$.

6.6 Exercises

Exercises 66 to 72 are devoted to the word case, which is out of scoop of this book. For this reason, we give precise hints for them.

Exercise 66. *The class of rational transductions is closed under rational operations.* Hint: for closure under union, connect a new initial state to initial state with $(\varepsilon, \varepsilon)$-rules (parallel composition). For concatenation, connect by the same way final states of the first transducer to initial states of the second (serial composition). For iteration, connect final states to initial states (loop operation).

Exercise 67. *The class of rational transductions is not closed under intersection.* Hint: consider rational transductions $\{(a^n b^p, a^n) \mid n, p \in \mathbb{N}\}$ and $\{(a^n b^p, a^p) \mid n, p \in \mathbb{N}\}$.

Exercise 68. *Equivalence of rational transductions is undecidable.* Hint: Associate the transduction $T_P = \{(f(u), g(u)) \mid u \in \Sigma^+$ with each instance $P = (f, g)$ of the Post correspondance Problem such that T_P defines $\{(\Phi(m), \Psi(m)) \mid m \in \Sigma^*\}$. Consider *Diff* of example 52.2. *Diff* $\neq$ *Diff* $\cup T_P$ if and only if P satisfies Post property.

Exercise 69. *Equivalence of deterministic rational transductions is decidable.* Hint: design a pumping lemma to reduce the problem to a bounded one by suppression of loops (if difference of lengths between two transduced subwords is not bounded, two transducers cannot be equivalent).

Exercise 70. *Build a rational transducer equivalent to a bimorphism.* Hint: let $f(q) \rightarrow q'(f)$ a transition rule of L. If $\Phi(f) = \varepsilon$, introduce transduction rule $\varepsilon(q) \rightarrow q'(\Psi(f))$. If $\Phi(f) = a_0 \ldots a_n$, introduce new states $q_1, \ldots, q_n$ and transduction rules $a_0(q) \rightarrow q_1(\varepsilon), \ldots a_i(q_i) \rightarrow q_{i+1}(\varepsilon), \ldots a_n(q_n) \rightarrow q'(\Psi(f))$.

Exercise 71. *Build a bimorphism equivalent to a rational transducer.* Hint: consider the set Δ of transition rules as a new alphabet. We may speak of the first state q and the second state q' in a letter "$f(q) \rightarrow q'(m)$". The control language L is the set of words over this alphabet, such that (i) the first state of the first letter is initial (ii) the second state of the last letter is final (iii) in every two consecutive letters of a word, the first state of the second equals the second state of the first. We define Φ and Ψ by $\Phi(f(q)- > q'(m)) = f$ and $\Psi(f(q)- > q'(m)) = m$.

Exercise 72. *Homomorphism inversion and applications.* An homomorphism Φ is non-increasing if for every symbol a, $\Phi(a)$ is the empty word or a symbol.

1. For any morphism Φ, find a bimorphism (Φ', L, Ψ) equivalent to Φ^{-1}, with Φ' non-increasing, and such that furthermore Φ' is ε-free if Φ is ε-free. Hint: Φ^{-1} is equivalent to a transducer R (Exercise 70), and the output homomorphism Φ' associated to R as in Exercise 71 is non-increasing. Furthermore, if Φ is ε-free, R and Φ' are ε-free.
2. Let Φ and Ψ two homomorphism. If Φ is non-increasing, build a transducer equivalent to $\Psi \circ \Phi^{-1}$ (recall that this notation means that we apply Ψ before Φ^{-1}). Hint and remark: as Φ is non-increasing, Φ^{-1} satisfies the inverse homomorphism property $\Phi^{-1}(MM') = \Phi^{-1}(M)\Phi^{-1}(M')$ (for any pair of words or languages M and M'). This property can be used to do constructions "symbol by symbol". Here, it suffices that the transducer associates $\Phi^{-1}(\Psi(a))$ with a, for every symbol a of the domain of Ψ.
3. Application: prove that classes of regular and context-free languages are closed under bimorphisms (we admit that intersection of a regular language with a regular or context-free language, is respectively regular or context-free).

4. Other application: prove that bimorphisms are closed under composition. Hint : remark that for any application f and set E, $\{(x, f(x)) \mid f(x) \in E\} = \{(x, f(x)) \mid x \in f^{-1}(E)\}$.

Exercise 73. We identify words with trees over symbols of arity 1 or 0. Let relations $U = \{(f^n a, f^n a) \mid n \in \mathbb{N}\} \cup \{(f^n b, g^n b) \mid n \in \mathbb{N}\}$ and $D = \{(ff^n a, ff^n a) \mid n \in \mathbb{N}\} \cup \{(gf^n a, gf^n b) \mid n \in \mathbb{N}\}$. Prove that U is a deterministic linear complete bottom-up transduction but not a deterministic top-down transduction. Prove that D is a deterministic linear complete top-down transduction but not a deterministic bottom-up transduction.

Exercise 74. Prove point 3 of Comparison Theorem. Hint. Use rule-by-rule techniques as in Exercise 75.

Exercise 75. Prove Composition Theorem. Hints: Prove 1 and 2 using composition "rule-by-rule", illustrated as following. States of $A \circ B$ are products of states of A and states of B. Let $f(q(x)) \rightarrow_A q'(g(x, g(x, a)))$ and $g(q_1(x), g(q_2(y), a) \rightarrow_B q_4(u)$. Subterms substituted to x and y in the composition must be equal, and determinism implies $q_1 = q_2$. Then we build new rule $f((q, q1)(x)) \rightarrow_{A \circ B} (q', q_4)(u)$. To prove 3 for example, associate $q(g(x, y)) \rightarrow u(q'(x), q''(y))$ with $g(q'(x), q''(y)) \rightarrow q(u)$, and conversely. For 4, Using ad hoc kinds of "rule-by-rule" constructions, prove $\mathsf{DDTT} \subset \mathsf{DCDTT} \circ \mathsf{LHOM}$ and $\mathsf{LHOM} \circ \mathsf{DCDTT} \subset \mathsf{DCDTT} \circ \mathsf{LHOM}$ (L means linear, C complete, D deterministic - and suffix DTT means top-down tree transducer as usually).

Exercise 76. Prove $\mathsf{NDTT} = \mathsf{HOM} \circ \mathsf{NLDTT}$ and $\mathsf{NUTT} = \mathsf{HOM} \circ \mathsf{NLBTT}$. Hint: to prove $\mathsf{NDTT} \subset \mathsf{HOM} \circ \mathsf{NLDTT}$ use a homomorphism H to produce in advance as may copies of subtrees of the input tree as the NDTT may need, ant then simulate it by a linear NDTT.

Exercise 77. Use constructions of composition theorem to reduce the number of passes in process of Section 6.3.

Exercise 78. Prove recognizability theorem. Hint: as in exercise 75, "naive" constructions work.

Exercise 79. Prove Theorem 49. Hint: "naive" constructions work.

Exercise 80. Prove point 2 of Theorem 50. Hint: E denote the class of homomorphisms which are linear and symbol-to-symbol. L, LC, LCF denotes linear, linear complete, linear complete ε-free homomorphisms, respectively. Prove $\mathsf{LCS} = \mathsf{L} \circ \mathsf{E} = \mathsf{E} \circ \mathsf{L}$ and $\mathsf{E}^{-1} \circ \mathsf{L} \subset \mathsf{L} \circ \mathsf{E}^{-1}$. Deduce from these properties and from point 1 of Theorem 50 that $\mathsf{LB}^4 = \mathsf{E} \circ \mathsf{LCFB}^2 \circ \mathsf{E}^{-1}$. To prove that $\mathsf{LB}^3 \neq \mathsf{LB}^4$, consider $\Psi_1 \circ \Psi_2^{-1} \circ \Phi \circ \Phi^{-1} \circ \Psi_2 \circ \Psi_1^{-1}$, where Φ is the homomorphism used in point 1 of Theorem 50; Ψ_1 identity on a, $f(x)$, $g(x, y)$, $h(x, y, z)$, $\Psi_1(e(x)) = x$; Ψ_2 identity on a, $f(x)$, $g(x, y)$ and $\Psi_2(c(x, y, z) = b(b(x, y), z)$.

Exercise 81. Sketch of proof of $\mathsf{LCFB}^2 = \mathsf{LCFB}^3$ (difficult). Distance $D(x, y, u)$ of two nodes x and y in a tree u is the sum of the lengths of two branches which join x and y to their younger common ancestor in u. $D(x, u)$ denotes the distance of x to the root of u.

Let H the class of deterministic top-down transducers T defined as follows: $q_0, \ldots, q_n$ are states of the transducer, q_0 is the initial state. For every context, consider the result u_i of the run starting from $q_i(u)$. $\exists k, \forall$ context u such that for every variable x of u, $D(x, u) > k$:

- u_0 contains at least an occurrence of each variable of u,

- for any i, u_i contains at least a non variable symbol,
- if two occurrences x' and x'' of a same variable x occur in u_i, $D(x', x", u_i) < k$.

Remark that LCF is included in H and that there is no right hand side of rule with two occurrences of the same variable associated with the same state. Prove that

1. $\mathsf{LCF}^{-1} \subseteq \mathsf{Delabeling}^{-1} \circ H$
2. $H \circ \mathsf{Delabeling}^{-1} \subseteq \mathsf{Delabeling}^{-1} \circ H$
3. $H \subseteq \mathsf{LCFB}^2$
4. Conclude. Compare with Exercise 72

Exercise 82. Prove that the image of a recognizable tree language by a linear tree transducer is recognizable.

6.7 Bibliographic notes

First of all, let us precise that several surveys have been devoted (at least in part) to tree transducers for 25 years. J.W. Thatcher [Tha73], one of the main pioneer, did the first one in 1973, and F. Gécseg and M. Steinby the last one in 1996 [GS96]. Transducers are formally studied too in the book of F. Gécseg and M. Steinby [GS84] and in the survey of J.-C. Raoult [Rao92]. Survey of M. Dauchet and S. Tison [DT92] develops links with homomorphisms.

In section 6.2, some examples are inspired by the old survey of Thatcher, because seminal motivation remain, namely modelization of compilers or, more generally, of syntax directed transformations as interfacing softwares, which are always up to date. Among main precursors, we can distinguish Thatcher [Tha73], W.S. Brainerd [Bra69], A. Aho, J.D. Ullman [AU71], M. A. Arbib, E. G. Manes [AM78]. First approaches where very linked to practice of compilation, and in some way, present tree transducers are evolutions of generalized syntax directed translations (B.S. Backer [Bak78] for example), which translate trees into strings. But crucial role of tree structure have increased later.

Many generalizations have been introduced, for example generalized finite state transformations which generalize both the top-down and the bottom-up tree transducers (J. Engelfriet [Eng77]); modular tree transducers (H. Vogler [EV91]); synchronized tree automata (K. Salomaa [Sal94]); alternating tree automata (G.Slutzki [Slu85]); deterministic top-down tree transducers with iterated look-ahead (G. Slutzki, S. Vàgvölgyi [SV95]). Ground tree transducers GTT are studied in Chapter 3 of this book. The first and the most natural generalization was introduction of top-down tree transducers with look-ahead. We have seen that "check and delete" property is specific to bottom-up tree transducers, and that missing of this property in the non-complete top-down case induces non closure under composition, even in the linear case (see Composition Theorem). Top-down transducers with regular look-ahead are able to recognize before the application of a rule at a node of an input tree whether the subtree at a son of this node belongs to a given recognizable tree language. This definition remains simple and gives to top-down transducers a property equivalent to "check and delete".

Contribution of Engelfriet to the theory of tree transducers is important, especially for composition, decomposition and hierarchy main results ([Eng75, Eng78, Eng82]).

We did not many discuss complexity and decidability in this chapter, because the situation is classical. Since many problems are undecidable in the word case, they are obviously undecidable in the tree case. Equivalence decidability holds as in the word case for deterministic or finite-valued tree transducers (Z. Zachar [Zac79], Z. Esik [Esi83], H. Seidl [Sei92, Sei94a]).

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