1. (a) Strategy space allows each agent to submit a bid (for the buyer, interpreted as maximum willingness to pay and for the seller, interpreted as minimum payment required.) Let $b_1, b_2 \geq 0$ denote bids for seller and buyer respectively. Trade if and only if $b_2 \geq b_1$, with payments $-b_2$ and $b_1$ respectively.

(b) To see VCG in double auction can run at a deficit, consider $v_1 = 5$, $v_2 = 10$. Then total deficit will be $-10 + 5 = -5$.

2. (a) Algorithm: sort bids in order of per-click price and allocate in decreasing order of bid.

(b) To determine per-click price compute the marginal externality imposed on other bidders and then divide by expected number of clicks in the slot allocated. For bidder 1,

$$
\text{payment}_1 = (p_1v_2 + \gamma p_1v_3 + \gamma^2 p_1v_4) - (\gamma p_1v_2 + \gamma^2 p_1v_3) \\
= p_1(1 - \gamma)v_2 + p_1(\gamma - \gamma^2)v_3 + p_1\gamma^2v_4
$$

Dividing by $p_1$, the expected number of clicks to bidder 1, the per-click price is

$$p_{\text{vcg},1} = (1 - \gamma)v_2 + (\gamma - \gamma^2)v_3 + \gamma^2v_4$$

For bidder 2,

$$
\text{payment}_2 = (p_1v_1 + \gamma p_1v_3 + \gamma^2 p_1v_4) - (p_1v_1 + \gamma^2 p_1v_3) \\
= p_1(\gamma - \gamma^2)v_3 + p_1\gamma^2v_4
$$

Dividing by $\gamma p_1$, the expected number of clicks to bidder 2, the per-click price is

$$p_{\text{vcg},2} = (1 - \gamma)v_3 + \gamma v_4$$
For bidder 3,
\[ \text{payment}_3 = (p_1 v_1 + \gamma p_1 v_2 + \gamma^2 p_1 v_4) - (p_1 v_1 + \gamma p_1 v_2) \]
\[ = p_1 \gamma^2 v_4 \]

Dividing by \( \gamma^2 p_1 \), the expected number of clicks to bidder 3, the per-click price is
\[ p_{vcg,3} = v_4 \]

(c) The per-click price in GSP is \( v_2, v_3 \) and \( v_4 \) to bidders 1, 2 and 3 respectively. For bidder 1, notice
\[ p_{vcg,1} = (1 - \gamma) v_2 + (\gamma - \gamma^2) v_3 + \gamma^2 v_4 \]
\[ < (1 - \gamma) v_2 + (\gamma - \gamma^2) v_2 + \gamma^2 v_2 \]
\[ = v_2 = p_{gsp,1}, \]

where the second equation follows if \( v_2 < v_3 < v_4 \) and since \( \gamma \in (0, 1) \) and thus \( \gamma - \gamma^2 > 0 \) and \( \gamma^2 > 0 \). A similar analysis holds for bidder 2. For bidder 3, the per-click price is the same. We conclude that the GSP price is strictly greater than the VCG price for all bidders except that allocated the last slot. [Of course, this might well change in equilibrium!]

3. Fix \( v_{-i}, v_i \) and let \( a = f(v) \in \arg \max_a c_a + \sum_i w_i v_i(a) \). Suppose some \( v'_i \neq v_i \) for which \( a' = f(v'_i, v_{-i}) \neq a \) and (ignoring the \( h_i(v_{-i}) \) which is independent of agent \( i \)'s bid),
\[ v_i(a') + \sum_{j \neq i} w_j \frac{v_j(a') - c_{a'}}{w_i} > v_i(a) + \sum_{j \neq i} w_j \frac{v_j(a) - c_a}{w_i} \]
\[ \Leftrightarrow w_i v_i(a') + \sum_{j \neq i} w_j v_j(a') - c_{a'} > w_i v_i(a) + \sum_{j \neq i} w_j v_j(a) - c_a, \]

which is a contradiction because \( a \) maximizes the RHS over all \( a'' \in A \).

4. (a) To show that bid \( b_i(v_i) = v_i \) is a (weak) dominant strategy in a single-item Vickrey auction proceed by case analysis. Let \( b_2 \) denote the highest bid across other bidders. (Case 1) If \( b_2 > v_1 \) then \( b_1(v_1) = v_1 \) is a best-response because agent 1 would rather lose than pay \( b_2 > v_1 \). (Case 2) If \( b_2 < v_1 \) then \( b_1(v_1) = v_1 \) is a best-response because agent 1 would rather win than pay \( b_2 < v_1 \). (Case 3) For \( b_2 = v_1 \) the agent is indifferent between losing and winning.

(b) Bidding truthfully remains a (weak) dominant strategy for the bidder because the reserve price can be treated just as an additional bid (and it is set before the bids are received.)
(c) Value $v_1 \sim U(0, \bar{v})$. Seller’s value is $v_0$. For the buyer, truthful bidding remains a (weak) dominant strategy just as above. For the seller, expected value given bid $r$ is

$$\Pr[v_1 \geq r]r + \Pr[v_1 < r]v_0 = \frac{\bar{v} - r}{\bar{v}} + \frac{r}{\bar{v}}v_0$$

Taking the derivative with respect to $r$ and setting to zero,

$$\frac{\bar{v} - r}{\bar{v}}(1) - \frac{1}{\bar{v}}r + \frac{v_0}{\bar{v}} = 0 \quad \Leftrightarrow \quad 1 - \frac{2r}{\bar{v}} + \frac{v_0}{\bar{v}} = 0 \quad \Leftrightarrow \quad r = \frac{\bar{v} + v_0}{2}$$

(d) Fix $\bar{v} = 1$, consider 2 bidders. Reserve price is $1/2$. To determine expected revenue consider 2 cases. Let $A$ denote event $v_1 > 1/2$ and $B$ denote event $v_2 > 1/2$. (Case 1). Both above 1/2. Revenue $Pr(A \land B)E[v(2)|v(2) \geq 1/2] = (1/4) \cdot (2/3) = 1/6$. (Case 2) Only one above 1/2. Revenue $Pr((A \land \neg B) \lor (B \land \neg A))/2 = (1/2) \cdot (1/2) = 1/4$. Expected revenue is $5/12$.

(e) Without a reserve price, $E[v(2)] = 0 + (1/3)(1) = 1/3$ by formula given (setting $k = 2, n = 2$.) The revenue is $1/12$ with the reserve price than without. The reserve price reduces efficiency because sometimes the item is not sold (i.e., when both bids are less than 1/2). We see a tradeoff between revenue and efficiency.

6. (a) Bids $\{(AB, 2), (A, 2), (B, 2)\}$ from 3 different bidders. VCG mechanism allocates to bidders 1 and 2, for zero payment each.

(b) To see failure of monotonicity, suppose that initially the bids are $\{(AB, 2), (A, 2)\}$. Allocate to bidder 1 (or bidder 2, breaking ties at random); collect revenue 2.

(c) To see vulnerability to failure, suppose bidders 2 and 3 have true values of 0.5 each. Will lose. But can collude with bids of 2 each and win for zero payment!

(d) To see false-name bid vulnerability, suppose bidder 2 wants $AB$ for value 3. Would win, and pay 2. Could split into two bids $\{(A, 2), (B, 2)\}$ and would win and pay 0!

(e) No, the single-item Vickrey auction is not susceptible in to these problems. For revenue monotonicity, an additional bidder can only increase $v^{(2)}$, the second-order statistic. For collusion vulnerability, any losing bidders must certainly pay more than the current winner and so they cannot deviate and benefit from winning. For false-name bid proofness, additional bids from the same bidder can only increase the price the bidder faces.