1. (a) Suppose otherwise, i.e. there is some \( v'_i \neq v_i \) for which \( v_i(x') - p(x') > v_i(x) - p(x) \) where \( x = x_i(v_i, v_{-i}) \) and \( x' = x_i(v'_i, v_{-i}) \) and \( p(x') = p(x', v_{-i}) \), \( p(x) = p(x, v_{-i}) \). But \( x = x_i(v) = \arg \max_{x \in X} [v_i(x) - p(x, v_{-i})] \) and this is a contradiction.

(b) For a Vickrey auction, define \( p_i(\text{lose}, v_{-i}) = 0 \), \( p_i(\text{win}, v_{-i}) = \max_{j \neq i} v_j \), and to establish (A2) see that if \( v_i > \max_{j \neq i} v_j \) then the agent wins and if \( v_i < \max_{j \neq i} v_j \) then the agent loses.

(c) (extra credit). Let \( L \subseteq G \) denote the bundle allocated to agent \( i \). Define agent-independent payment \( p_i(L) = V_{-i}(G) - V_{-i}(G \setminus L) \) where \( V_{-i}(T) \) for goods \( T \subseteq G \) is the value of the maximal allocation of goods \( T \) to agents \( \neq i \). To show that the efficient allocation \( S^* \in \arg \max_S \sum_i v_i(S_i) \) satisfies A2, see that the bundle that satisfies A2 is

\[
\arg \max_S \left[ v_i(S) + V_{-i}(G \setminus S) - V_{-i}(G) \right] = \arg \max_S \left[ v_i(S) + V_{-i}(G \setminus S) \right],
\]

which is true for \( S^*_i \) because this optimization problem is equivalent to maximizing the total value over all agents. To see that the payment in VCG is equal to the price \( p_i(L) = V_{-i}(G) - V_{-i}(G \setminus L) \), observe that agent \( i \) pays

\[
\sum_{j \neq i} v_j(S'_j) - \sum_{j \neq i} v_j(S^*_j) = V_{-i}(G) - V_{-i}(G \setminus S^*_i),
\]

where \( S' \) is the allocation that solves \( \max_S \sum_{j \neq i} v_j(S_j) \) and the the second terms are equal by the principle of optimality. (The allocation to agents \( \neq j \) must be optimal, given that \( S^*_i \) is allocated to agent \( i \).)
2. Fix $v_{-i}$, and consider $f(v_i) = a \neq b = f(v'_i)$. WMON requires that $v'_i(b) - v'_i(a) \geq v_i(b) - v_i(a)$. So, if $f(v_i) = a \in W_i$ and $f(v'_i) = b \notin W_i$, then WMON requires that $-v'_i \geq -v_i$ and $v'_i \leq v_i$. Since $v'_i \geq v_i$ we must have $v'_i = v_i$, but then $a = b$ and a contradiction.

4. (a) To see that myopic best-response (MBR) is not a dominant-strategy for bidder 1 with value $50$, suppose bidder 2 has a “crazy” strategy such as

“If the ask price is ever less than $5$ then bid $100$, else bid at the ask price while losing and the ask price is no greater than $10$.”

Now if bidder 1 bids MBR, then the price will initially be less than $5$ and this will trigger the crazy response of bidder 2, and thus bidder 1 will lose. Bidder 1 is better off submitting a jump bid to $5 + \epsilon$, so that she will eventually win for a price around $10$.

(b) The question asked for a solution concept in which MBR is an equilibrium. In fact, we need to be a bit more careful.

Even if we make the price increment in the auction $1$, and require that bids and agent values are multiples of $1$, MBR is not even an \textit{ex post} NE. To see this, suppose $v_1 = 50, v_2 = 10$. Following MBR could lead to agent 1 winning with a bid of $9$, ask price of $10$. Now agent 2 would bid (but not agent 1). Agent 2 is winning, ask price is $11$. Finally agent 1 bids and wins for $11$. But agent 1 could jump bid to $10$ and win for $10$.

It can be shown that MBR is an approximate \textit{ex post} NE, with bid increment $\epsilon$ and all bids and values multiples of $\epsilon$ then MBR is an $\epsilon$ \textit{ex post} NE in that every agent that follows MBR gets within $\epsilon$ of its best-possible utility for all value profiles as long as the other agents also follow MBR.

(Not required.) For a proof sketch, consider agent $i$ and let $v_2 = \max_{j \neq i} v_j$.

\textbf{Lemma 1.} Agent $i$ cannot win for a price $\leq v_2 - \epsilon$, for any strategy, as long as agents $\neq i$ follow MBR strategy.

\textit{Proof.} Suppose otherwise,... then the auction must close with agents $\neq i$ not winning, and the ask price $\leq v_2$. But then at least one agent $\neq i$ would bid in the MBR strategy. \hfill $\Box$

\textbf{Lemma 2.} If agent $i$ follows the MBR strategy, then it cannot win for a price $> v_2 + \epsilon$.

\textit{Proof.} This would require the bids of other agents to be $> v_2$, which is not the case when they follow a MBR strategy. \hfill $\Box$

\textbf{Lemma 3.} If agent $i$ follows the MBR strategy, then it will definitely win if $v_i > v_2$. 

2
Proof. Suppose otherwise, then the auction would close with ask price \( v_1 \), and thus ask price \( \geq v_1 + \epsilon \), and thus another bidder’s winning bid price \( \geq v_1 \). But this cannot be since \( v_1 > v_2 \). \qed

Theorem 1. MBR is an \( \epsilon \)-ex post Nash equilibrium of the ascending-price auction with jump bids.

Proof. Fix \( v_1 \) and the values of other bidders. Let \( v_2 \) denote the highest value of other bidders. By case analysis. Note that utility to agent 1 for MBR strategy is \( \geq 0 \).

(Case 1). \( v_1 \leq v_2 \). By Lemma 1 and with values and bids multiples of \( \epsilon \), if agent 1 wins then the price is \( \geq v_2 \). Then maximal utility from any strategy is \( v_1 - v_2 \), which is non-positive. Thus the MBR strategy is just as good.

(Case 2). \( v_1 > v_2 \). By Lemma 1, maximal utility from any strategy is \( v_1 - v_2 \), but MBR strategy will definitely win by Lemma 3 and with utility at least \( v_1 - (v_2 + \epsilon) \) by Lemma 2. \qed

Without the requirement about bids and values being multiples of the bid increment \( \epsilon \) the proof could be modified to achieve a \( 2\epsilon \) epNE.

(c) Consider a clock auction without jump bids. Bidding in the straightforward way (keeping arm raised until price = true value, then dropping) is a dominant strategy here because conditioned on winning, all other strategies (with a drop-out point higher than the closing price) are equivalent as viewed by the other agents and thus do not affect in anyway the strategies of other agents. It is also easy to see that if an agent wins with the straightforward strategy it cannot do better by losing.

5. (a) For agent 1, \( \phi_1(v_1) = v_1 - (1 - v_1)/1 = 2v_1 - 1 \). For agent 2, \( \phi_2(v_2) = v_2 - (1 - v_2)/2)/(1/2) = 2v_2 - 2 \). The virtual valuation functions are clearly monotone non-decreasing.

(b) To gain some intuition for why these (asymmetric) virtual valuations help to improve revenue by making the auction more competitive, consider the following observations:

(i) with virtual valuations, a value of between 1 and 1.5 from bidder 2 competes with a value of between 0.5 and 1 from bidder 1, whereas in the efficient auction bidder 2 always wins with a value \( \geq 1 \).

(ii) with virtual valuations, bids less than 1 from bidder 2 cannot win, whereas in the efficient auction bidder 2 can still compete with values between 0.5 and 1 with values of 0.5 to 1 from bidder 1.

(iii) if bidder 2 wins, his payment is necessarily more than the bid of bidder 1 (since his payment is always at least 1).
6. (thanks to Hajir and Dong for much of this...)

(a) The hazard rate of the value distribution is \(1/(2 - v_i)\) for \(v_i \in [0, 1)\) and \(1/(3 - v_i)\) for \(v_i \in [1, 3]\) and is not monotone non-decreasing. The virtual valuations are

\[
\phi(v_i) = \begin{cases} 
2v_i - 2 & \text{if } 0 \leq v_i < 1 \\
2v_i - 3 & \text{if } 1 \leq v_i \leq 3
\end{cases}
\]

and also not monotone non-decreasing.

(b) We have

\[
F^{-1}(x) = \begin{cases} 
2x & \text{if } x \in [0, 1/2) \\
4x - 1 & \text{if } x \in [1/2, 1]
\end{cases}
\]

Thus,

\[
\gamma(q) = \phi(F^{-1}(q)) = \begin{cases} 
4q - 2 & \text{if } q \in [0, 1/2) \\
8q - 5 & \text{if } q \in [1/2, 1]
\end{cases}
\]

To integrate the function with respect to \(q\),

\[
\beta(q) = \begin{cases} 
\int_0^q (4t - 2) dt & \text{if } q \in [0, 1/2) \\
\int_{1/2}^q (8t - 5) dt & \text{if } q \in [1/2, 1]
\end{cases}
\]

\[
= \begin{cases} 
2q^2 - 2q & \text{if } q \in [0, 1/2) \\
4q^2 - 5q + 1 & \text{if } q \in [1/2, 1]
\end{cases}
\]

To construct the convex lower hull, assume that there is a straight-line tangent between the two curves. Assume: \(y = aq + b\) defines the tangent, and let \(m\) and \(n\) correspond to the \(q\)-values of the two points at which the tangent intersects the curve 1 and 2 respectively. We have

\[
\begin{align*}
a &= 4n - 2 \\
a &= 8m - 5 \\
am + b &= 2n^2 - 2n \\
am + b &= 4m^2 - 5m + 1
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
a = 1 - \sqrt{2} \\
b = \frac{6\sqrt{2} - 11}{8} \\
c = \frac{3 - \sqrt{2}}{4} \\
m = \frac{\sqrt{2} - 2}{8}
\end{cases}
\]

so that the modified curve is

\[
\hat{\beta}(q) = \begin{cases} 
2q^2 - 2q & \text{if } q \in [0, 3 - \sqrt{2}) \\
(1 - \sqrt{2})q + \frac{6\sqrt{2} - 11}{8} & \text{if } q \in [3 - \sqrt{2}, 6 - \sqrt{2}) \\
4q^2 - 5q + 1 & \text{if } q \in [6 - \sqrt{2}, 1]
\end{cases}
\]

Differentiating yields,

\[
\hat{\phi}(q) = \begin{cases} 
4q - 2 & \text{if } q \in [0, 3 - \sqrt{2}) \\
1 - \sqrt{2} & \text{if } q \in [3 - \sqrt{2}, 6 - \sqrt{2}) \\
8q - 5 & \text{if } q \in [6 - \sqrt{2}, 1]
\end{cases}
\]
and finally, by substituting \( q = v/2 \) for \( q \in [0, 1/2) \) and \( q = (v + 1)/4 \) for \( q \in [1/2, 1] \) we obtain,

\[
\hat{\phi}(v) = \begin{cases} 
2v - 2 & , \text{if } v \in [0, \frac{3-\sqrt{2}}{2}) \\
1 - \sqrt{2} & , \text{if } v \in [\frac{3-\sqrt{2}}{2}, \frac{4-\sqrt{2}}{2}) \\
2v - 3 & , \text{if } v \in [\frac{4-\sqrt{2}}{2}, 3]
\end{cases}
\]

[These can all be graphed!]

7. (a) The SPSB is more susceptible to collusion than the FPSB because bidders colluding in the SPSB can submit bids \((v, 0, 0)\) where \( v \) is the maximum of their values while they need to bid more like \((\epsilon, 0, 0)\) in the FPSB. Notice that no agent in the coalition can usefully deviate in the SPSB but that an agent can deviate and break away from the coalition in the FPSB, and thus the coalition is not stable.

(b) False. A single eBay auction is not (strong) strategically equivalent to a SPSB auction. To see this, notice that the strategy of one bidder in eBay can convey information to another bidder, with the second bidder then being able to condition his or her strategy on this information, thus enabling new kinds of equilibria.

(c) Sniping can be rational on eBay despite its similarity to a SPSB auction for a number of different reasons, including (i) if another bidder is a “follower” and bids upwards whenever losing, then it is better to snipe to prevent this kind of competition; (ii) if there are multiple auctions then sniping allows a bidder to wait until the last minute so as to bid in the auction that has the lowest (expected) clearing price.

(d) No, you might want to bid less than your true value on Monday; e.g., my value is 10 and the highest bid from other bidders is 8 on Monday but 5 on Tuesday. Then I should bid less than 8 (or not at all) on Monday and bid 10 on Tuesday.