Extensive-Form Games with Perfect Information

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September 22, 2008

Logistics

In this unit, we cover 5.1 of the SLB book.

Problem Set 1, due Wednesday September 24 in class.

Review: Normal-Form Games

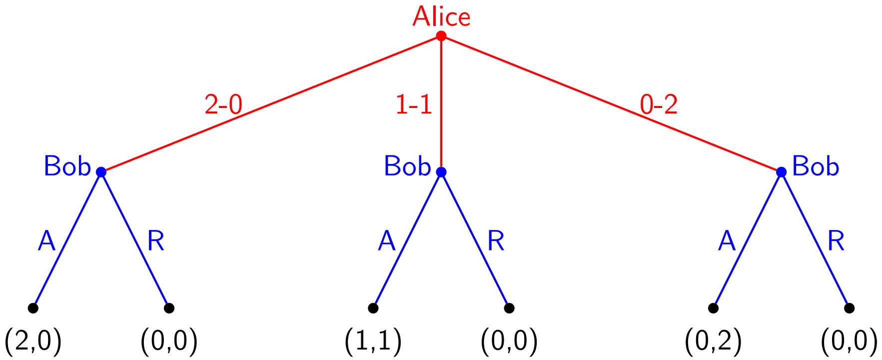
- A finite n-person normal-form game, $G =< N, A, u >$
  - $N = \{1, 2, ..., n\}$ is the set of players.
  - $A = \{A_1, A_2, ..., A_n\}$ is a set of available actions.
  - $u = \{u_1, u_2, ...u_n\}$ is a set of utility functions for n agents.

	C	D
C	5, 5	0, 8
D	8, 0	1, 1

Prisoner's Dilemma

Example 1: The Sharing Game

- Alice and Bob try to split two indivisible and identical gifts. First, Alice suggests a split: which can be "Alice keeps both", "they each keep one", and "Bob keeps both". Then, Bob chooses whether to Accept or Reject the split. If Bob accepts the split, they each get what the split specifies. If Bob rejects, they each get nothing.



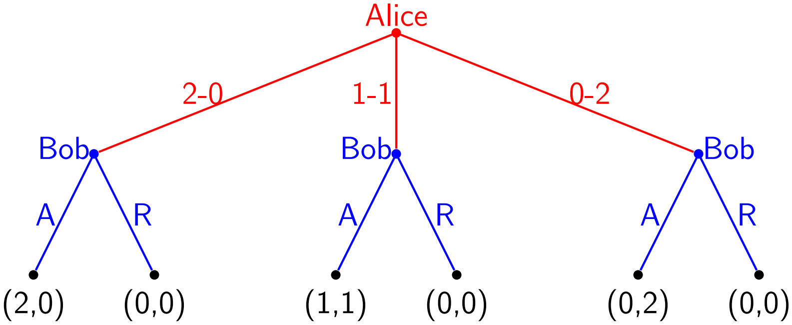
Loosely Speaking...

- Extensive Form
  - A detailed description of the sequential structure of the decision problems encountered by the players in a game.
  - Often represented as a game tree
- Perfect Information
  - All players know the game structure (including the payoff functions at every outcome).
  - Each player, when making any decision, is perfectly informed of all the events that have previously occurred.

Def. of Perfect-Information Extensive-Form Games

- A perfect-information extensive-form game, $G = (N, H, P, u)$
  - $N = \{1, 2, ..., n\}$ is the set of players.

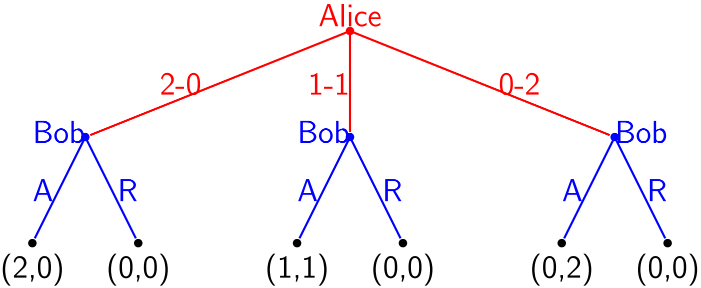
    N={Alice, Bob}



Def. of Perfect-Information Extensive-Form Games

- A perfect-information extensive-form game, $G = (N, H, P, u)$
  - H is a set of sequences (finite or infinite)
    - ⋆ $\Phi \in H$
    - ⋆ $h = (a^k)_{k=1,\ldots,K} \in H$ is a history
    - ⋆ If $(a^k)_{k=1,\ldots,K} \in H$ and $L < K$, then $(a^k)_{k=1,\ldots,L} \in H$
    - ⋆ $(a^k)_{k=1}^{\infty} \in H$ if $(a^k)_{k=1,\ldots,L} \in H$ for all positive L
    - ⋆ Z is the set of terminal histories.

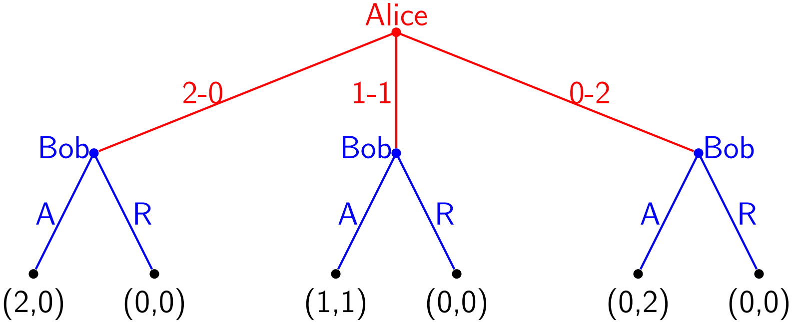
  $H = \{\Phi, 2-0, 1-1, 0-2, (2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\}$



Def. of Perfect-Information Extensive-Form Games

- A perfect-information extensive-form game, $G = (N, H, P, u)$
  - P is the player function, $P : H \setminus Z \rightarrow N$.

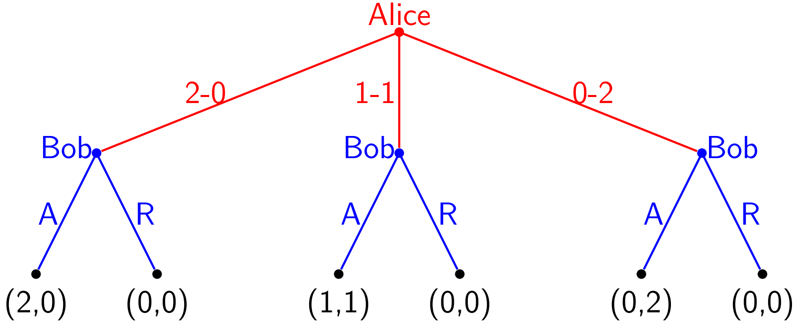
    $P(\Phi)$=Alice
    $P(2-0)$=Bob
    $P(1-1) =$ Bob
    $P(0-2) =$ Bob



Def. of Perfect-Information Extensive-Form Games

- A perfect-information extensive-form game, $G = (N, H, P, u)$
  - $u = \{u_1, u_2, ...u_n\}$ is a set of utility functions, $u_i : Z \rightarrow \mathcal{R}$.

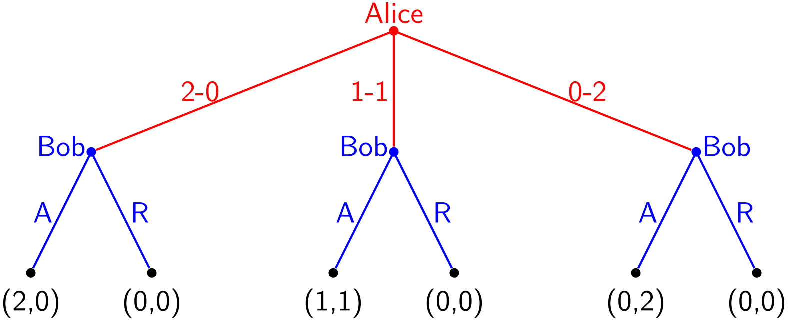
    $u_1((2-0, A)) = 2$
    $u_2((2-0, A)) = 0$
    ...



Pure Strategies in Perfect-Information Extensive-Form Games

- A pure strategy of player $i \in N$ in an extensive-form game with perfect information, $G = (N, H, P, u)$, is a function that assigns an action in $A(h)$ to each non-terminal history $h \in H \backslash Z$ for which $P(h) = i$.
  - $A(h) = \{a : (h, a) \in H\}$
  - A pure strategy is a contingent plan that specifies the action for player i at every decision node of i.

Pure Strategies for Example 1

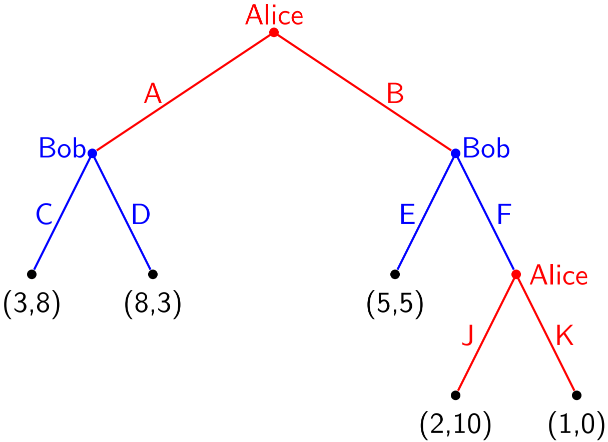


$S = \{S_1, S_2\}$

E.g. $s_1 = (2 - 0 \text{ if } h = \Phi)$,

$s_2 = (A \text{ if } h = 2 - 0; \text{ R if h = 1-1}; R \text{ if } h = 0 - 2)$.

Pure Strategies: Example 2



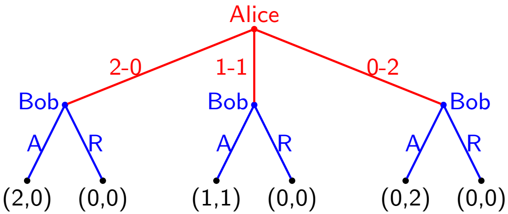
$S = \{S_1, S_2\}$

E.g. $s_1 = (A \text{ if } h = \Phi;\ J \text{ if } h = BF)$

$s_2 = (C \text{ if } h = A;\ F \text{ if } h = B)$

Normal-Form Representation: Example 1

A perfect-information extensive-form game ⇒ A normal-form game

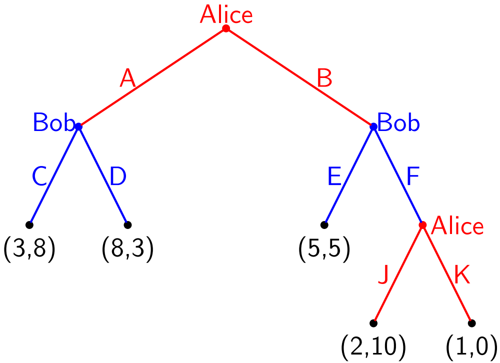


	(A,A,A)	(A,A,R)	(A,R,A)	(A,R,R)	(R,A,A)	(R,A,R)	(R,R,A)	(R,R,R)
2-0	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0
1-1	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
0-2	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0

A normal-form game ⇏ A perfect-information extensive-form game

Normal-Form Representation: Example 2

A perfect-information extensive-form game ⇒ A normal-form game

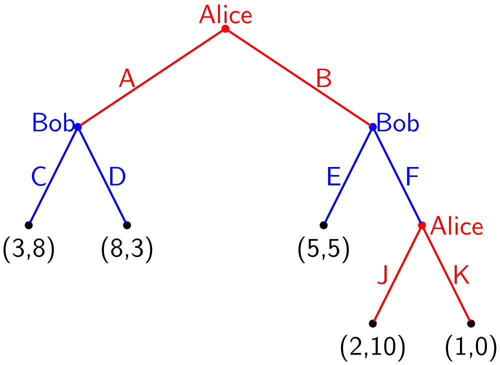


	(C, E)	(C, F)	(D, E)	(D, F)
(A, J)	3, 8	3, 8	8, 3	8, 3
(A, K)	3, 8	3, 8	8, 3	8, 3
(B, J)	5, 5	2,10	5, 5	1, 10
(B, K)	5, 5	1, 0	5, 5	1, 0

A normal-form game ⇏ A perfect-information extensive-form game

Pure Strategy Nash Equilibrium in Perfect-Information Extensive-Form Games

- A pure strategy profile s is a weak Nash Equilibrium if, for all agents i and for all strategies $s'_i \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.
  (Same as in normal-form games)



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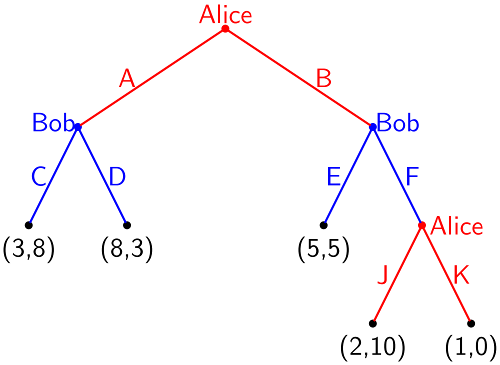

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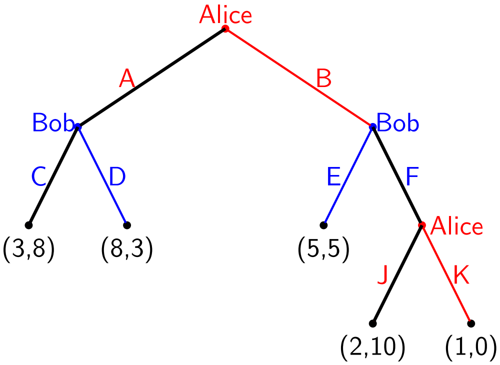

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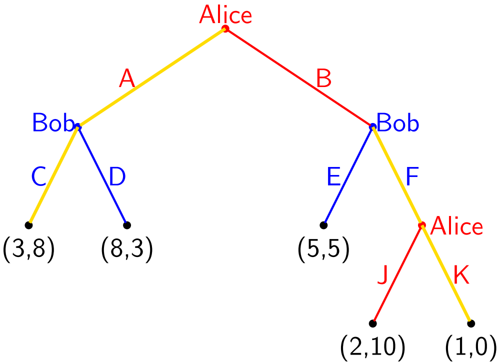

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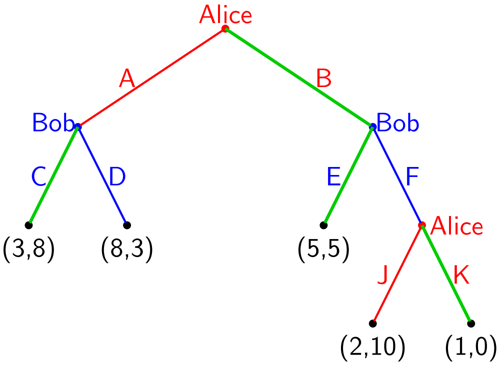

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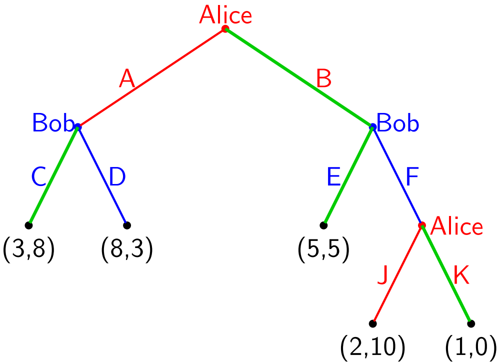
Nash Equilibrium and Non-Credible Threat

- Nash Equilibrium is not a very satisfactory solution concept for perfect-information extensive-form games.



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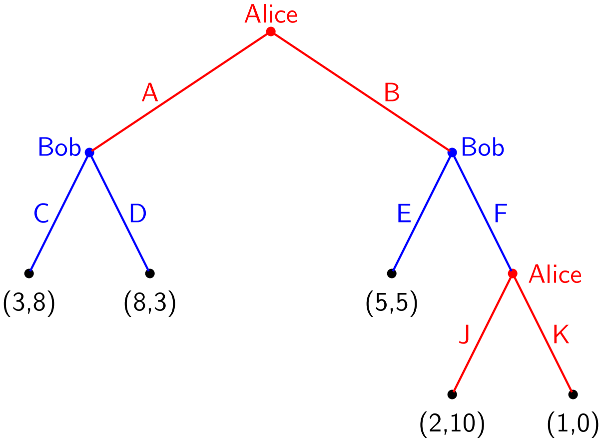
(B, K)	5, 5	1, 0	5, 5	1, 0

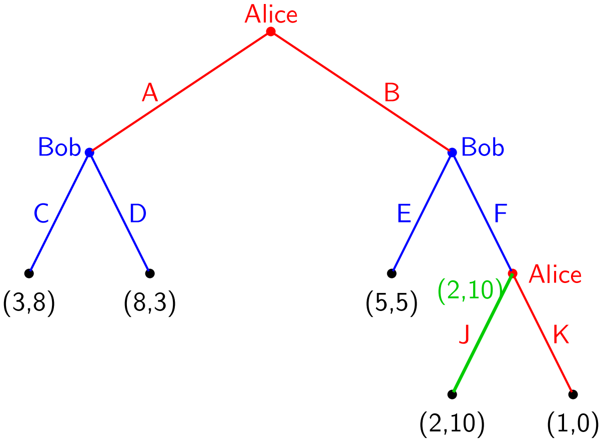
Subgame Perfect Equilibrium

- Sequential Rationality: A player's equilibrium strategy should specify optimal actions at every point in the game tree.
- A Subgame Perfect Equilibrium (SPE) of a perfect-information extensive-form game G is a strategy profile s such that for any subgame G' of G, the restriction of s to G' is a NE.
- Every SPE is a NE, but not vice versa.
- **Thm:** Every finite extensive-form game with perfect information has a subgame perfect equilibrium.
  - Finite: The set of sequences H is finite.

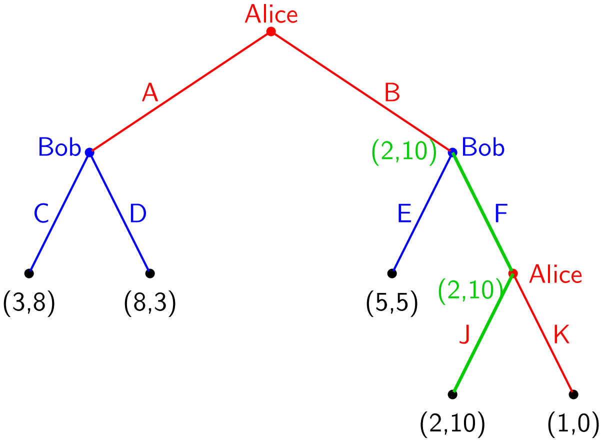
Find A SPE: Backward Induction



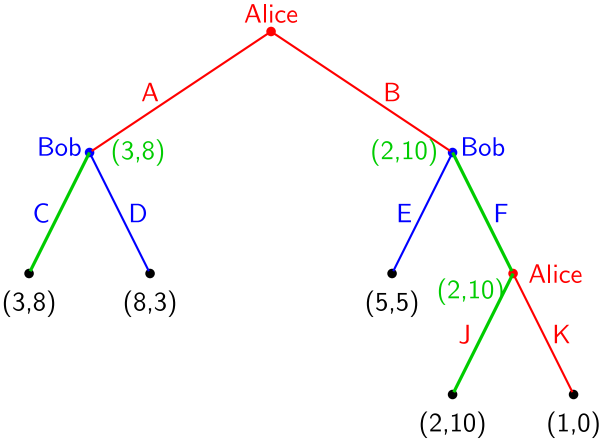
Find A SPE: Backward Induction



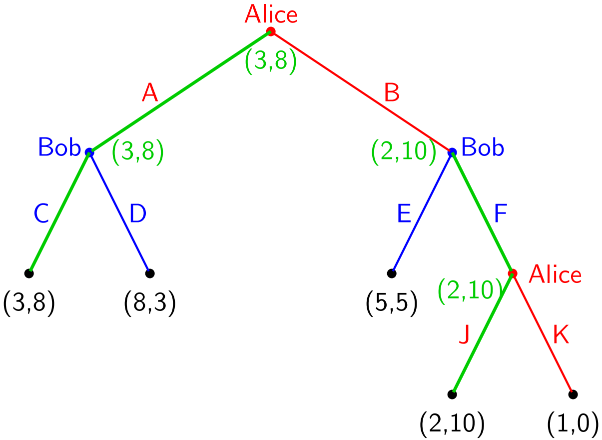
Find A SPE: Backward Induction



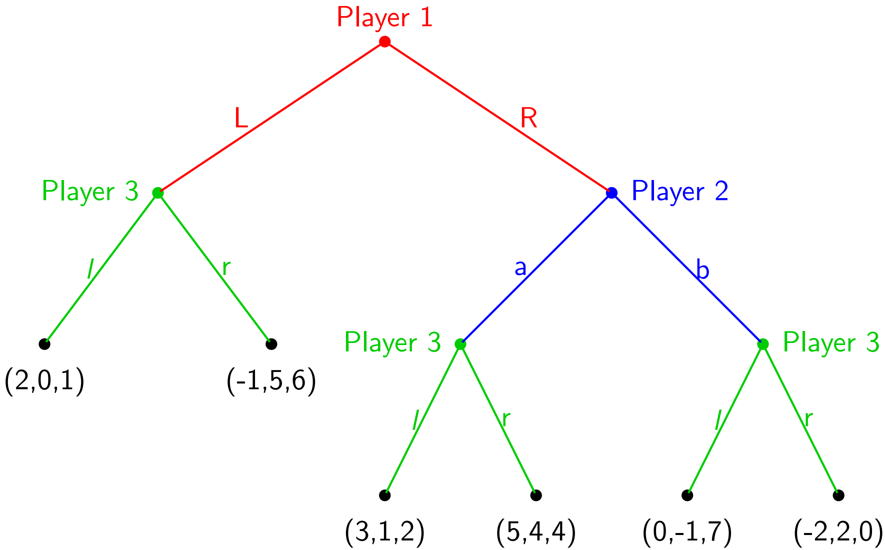
Find A SPE: Backward Induction



Find A SPE: Backward Induction



Example 3


Player 1
L
R
Player 3
Player 2
l
r
a
b
(2,0,1)
(-1,5,6)
Player 3
Player 3
l
r
l
r
(3,1,2)
(5,4,4)
(0,-1,7)
(-2,2,0)

Note on Computational Complexity

- Finding NE for general normal-form games requires time exponential in the size of the normal form.
- The induced normal form of an extensive-form game is exponentially larger than the original representation.
- Algorithm of backward induction requires time linear in the size of the extensive-form game. (Depth-first transverse)
- For zero-sum extensive-form games, we can slightly improve the running time.

A Bargaining Game: Split-the-Pie

- Two players trying to split a desirable pie. The set of all possible agreements X is the set of all divisions of the pie,

$$X = \{(x_1, x_2) : x_i \geq 0 \text{ for } i = 1, 2 \text{ and } x_1 + x_2 = 1\}.$$

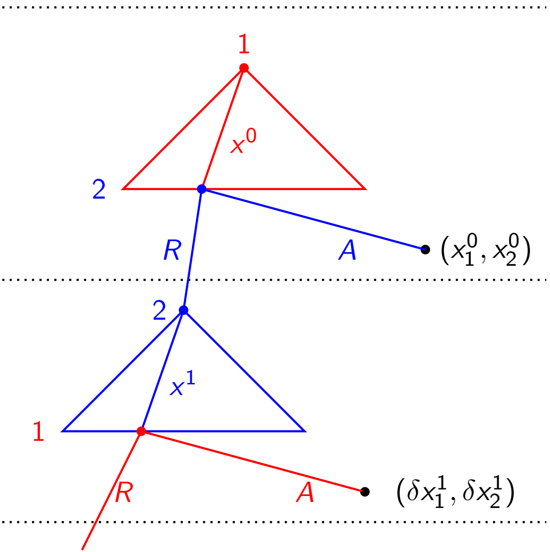
- The first move of the game occurs in period 0, when player 1 makes a proposal $x^0 \in X$, then player 2 either accepts or rejects. Acceptance ends the game while rejection leads to period 1, in which player 2 makes a proposal $x^1 \in X$, which player 1 has to accept or reject. Again, acceptance ends the game; rejection leads to period 2, in which it is once again player 1's turn to make a proposal. The game continues in this fashion so long as no offer has been accepted.
- $u_i(x, t) = \delta^t x_i$ if proposal x has been accepted in period t, $\delta \in (0, 1)$.
- $u_i = 0$ if no agreement has reached.

Split-the-Pie as A Perfect-Information Extensive-Form Game

$G = (N, H, P, u)$

- N: {Player 1, Player 2}
- H is the set of all sequences of one of the following
  - I: Φ, or $(x^0, R, x^1, R, ..., x^t, R)$
  - II: $(x^0, R, x^1, R, ..., x^t)$
  - III: $(x^0, R, x^1, R, ..., x^t, A)$
  - IV: $(x^0, R, x^1, R, ...)$
- P is the player function
  - $P(h) = 1$ if $h \in I$ or II and t is odd or $h = \Phi$
  - $P(h) = 2$ if $h \in I$ or II and t is even
- $u = (u_1, u_2)$ is the utility function
  - $u_i((x^0, R, x^1, R, ..., x^t, A)) = \delta^t x_i^t$
  - $u_i((x^0, R, x^1, R, ...)) = 0$

Split-the-Pie as A Perfect-Information Extensive-Form Game



Nash Equilibria of the Split-the-Pie Game

- The set of NEs is very large. For example, for any $x \in X$ there is a NE in which the players immediately agree on x.

  E.g. Player 1 always propose (0.99, 0.01) and only accepts a proposal (0.99, 0.01).

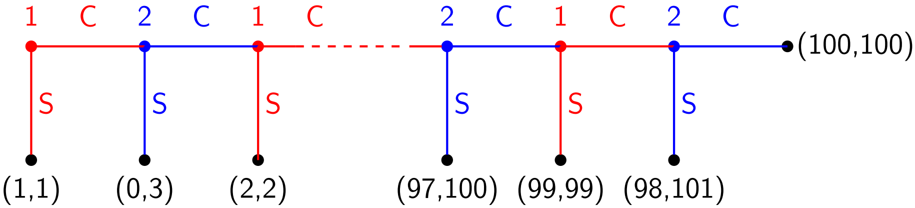
SPE of the Split-the-Pie Game

The unique SPE of the game is

- Player 1 always proposes $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$ and accepts proposals that has $x_1 \geq \frac{\delta}{1+\delta}$.
- Player 2 always proposes $(\frac{\delta}{1+\delta}, \frac{1}{1+\delta})$ and accepts proposals that has $x_2 \geq \frac{\delta}{1+\delta}$.

Centipede Game

- In this finite game of perfect information, there are two players, 1 and 2. The players each start with \$1 in front of them. They alternate saying "stop" or "continue", starting with player 1. When a player says "continue", \$1 is taken by a referee from her pile and \$2 are put in her opponent's pile. As soon as either player says "stop", play is terminated, and each player receives the money currently in her pile. Alternatively, play stops if both players' piles reach \$100.



Critique of SPE

- In experiments, subjects continue to play "continue" until the end of the game.
- If the second player observes that the first player chooses "continue", what should he do?

Summary

- Extensive-form games model the sequence of play.
- Every perfect-information extensive-form game has a induced normal-form game, but not vice versa.
- Nash equilibrium of perfect-information extensive-form can not deal with incredible threat.
- Subgame perfect equilibrium by introducing sequential rationality avoids incredible threat.
- Every finite perfect-information extensive-form has a SPE that can be found by backward reduction
- SPE has limitations on off-equilibrium paths