Extensive-Form Games
with Imperfect Information

Yiling Chen

September 12, 2011

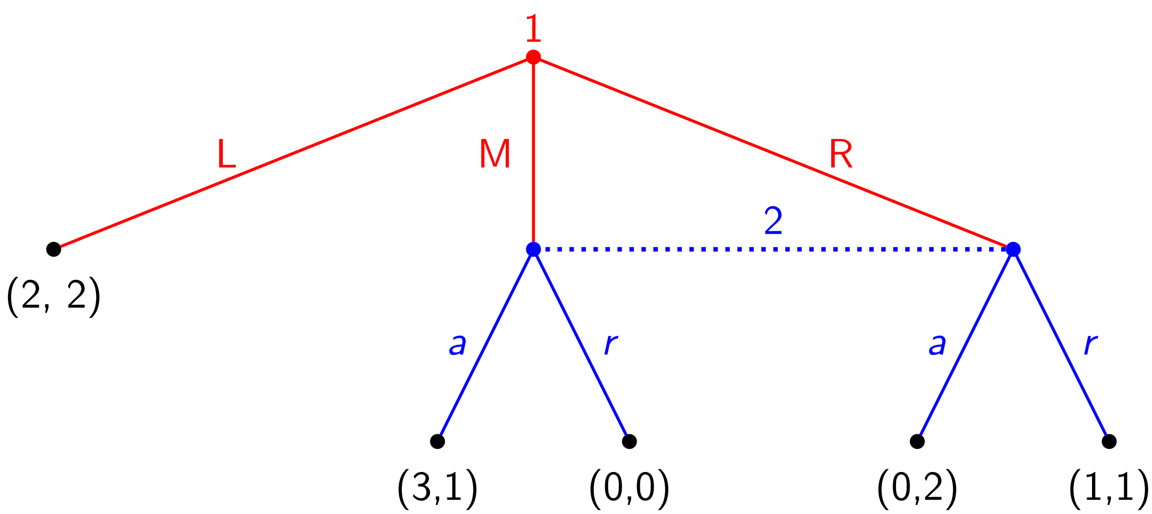
Perfect Information vs. Imperfect Information

- Perfect Information
  - All players know the game structure.
  - Each player, when making any decision, is perfectly informed of all the events that have previously occurred.
- Imperfect Information
  - All players know the game structure.
  - Each player, when making any decision, may not be perfectly informed about some (or all) of the events that have already occurred.

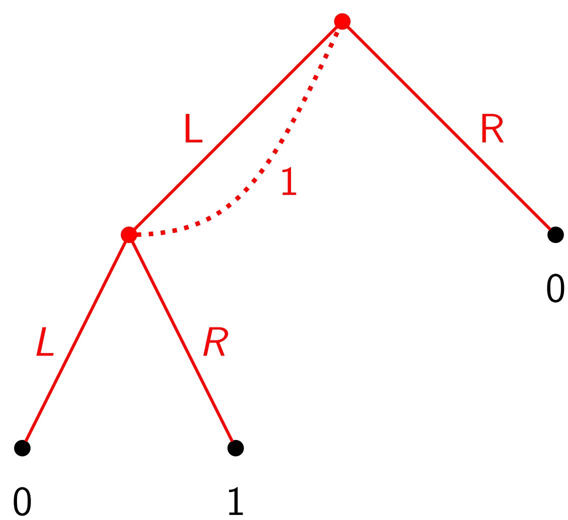
Roadmap

- Define Imperfect-Information Extensive-Form Game
- Introduce Sequential Equilibrium
  "rather a lot of bodies are buried in this definition". (Kreps, 1990)

Example 1



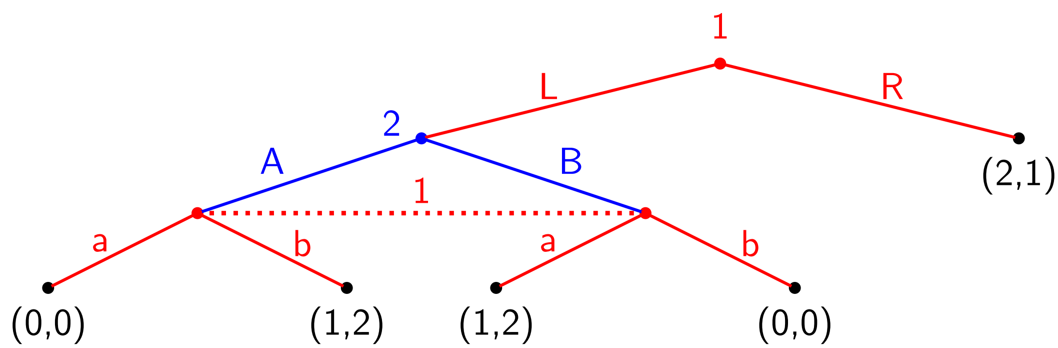
Example 2



Def. of Imperfect-Information Extensive-Form Games

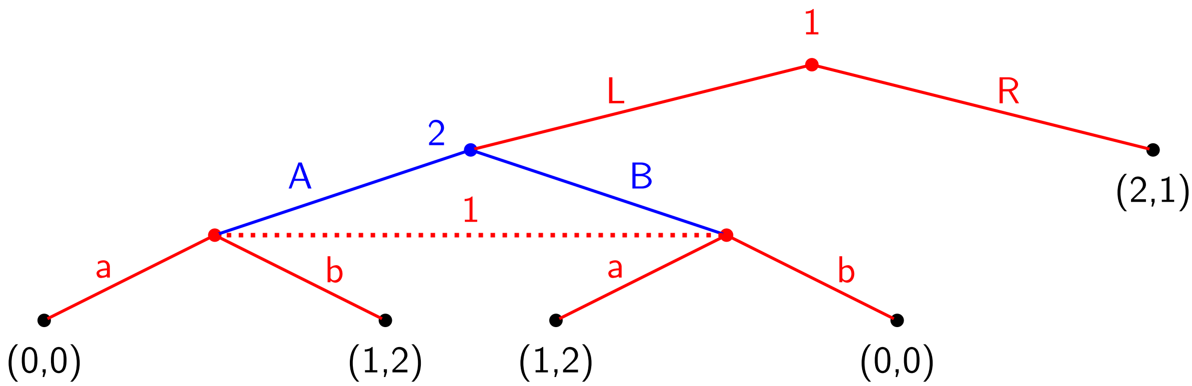
- An imperfect-information extensive-form game is a tuple $(N, H, P, \mathcal{I}, u)$
  - (N, H, P, u) is a perfect-information extensive-form game
  - $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, ..., \mathcal{I}_n\}$ is the set of information partitions of all players
    - $\mathcal{I}_i = \{\mathcal{I}_{i,1}, ...\mathcal{I}_{i,k_i}\}$ is the information partition of player i
    - $\mathcal{I}_{i,j}$ is an information set of player i
    - Action set $A(h) = A(h')$ if h and h' are in $\mathcal{I}_{i,j}$, denote as $A(\mathcal{I}_{i,j})$
    - $P(\mathcal{I}_{i,j})$ be the player who plays at information set $\mathcal{I}_{i,j}$.

  $\mathcal{I}_1 = \{\{\Phi\}, \{(L, A), (L, B)\}\}$, $\mathcal{I}_2 = \{\{L\}\}$



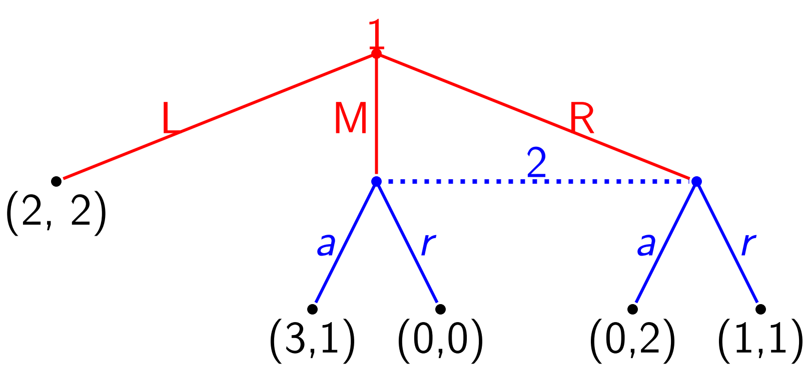
Pure Strategies: Example 3

- $S = \{S_1, S_2\}$
- $S_1 = \{(L, a), (L, b), (R, a), (R, b)\}$
- $S_2 = \{A, B\}$



Normal-Form Representation: Example 1

An imperfect-information extensive-form game ⇒ A normal-form game

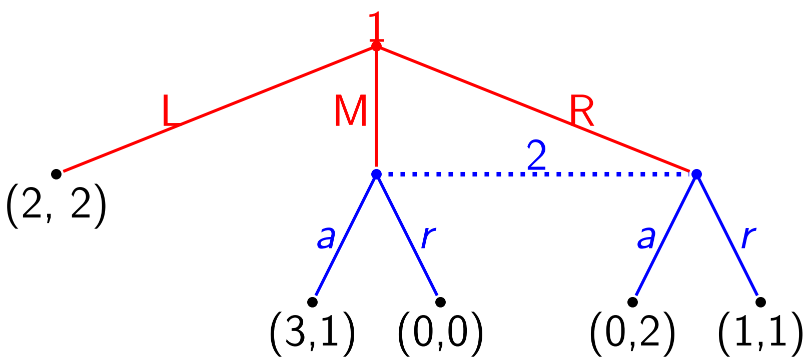


	a	r
L	2,2	2,2
M	3,1	0,0
R	0,2	1,1

The Nash Equilibrium (both pure and mixed) concept remains the same for imperfect-information extensive-form games.

Normal-Form Representation: Example 1

An imperfect-information extensive-form game ⇒ A normal-form game



	a	r
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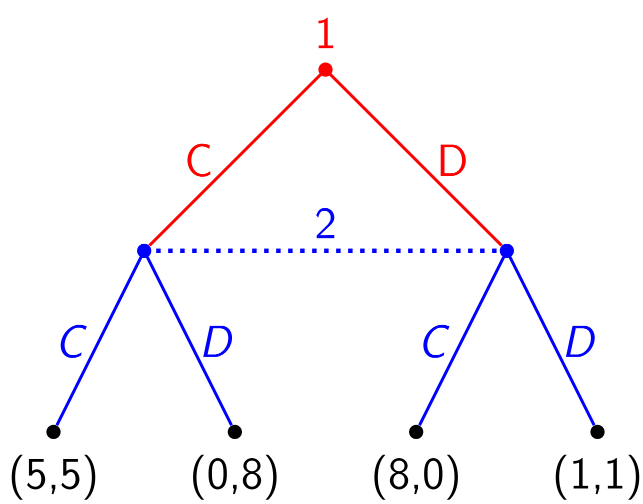
The Nash Equilibrium (both pure and mixed) concept remains the same for imperfect-information extensive-form games.

Normal-Form Games

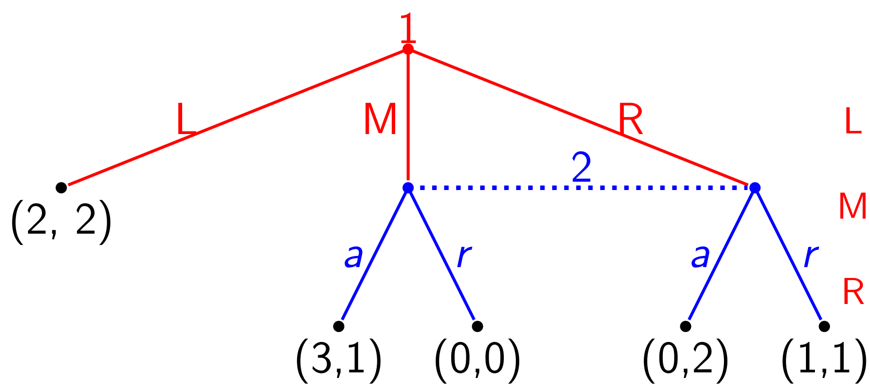
A normal-form game ⇒ An imperfect-information extensive-form game

	C	D
C	5, 5	0, 8
D	8, 0	1, 1

Prisoner's Dilemma



Nash Equilibrium: Example 1



	a	r
L	2,2	2,2
M	3,1	0,0
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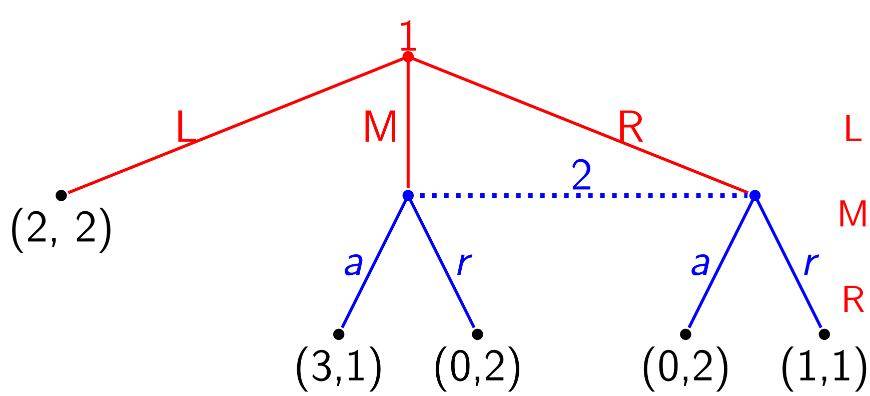
Suppose we want to generalize the idea of subgame perfect equilibrium. Consider the equilibrium (L, r). Is it subgame perfect?

Nash Equilibrium: Example 1



	a	r
L	2,2	2,2
M	3,1	0,2
R	0,2	1,1

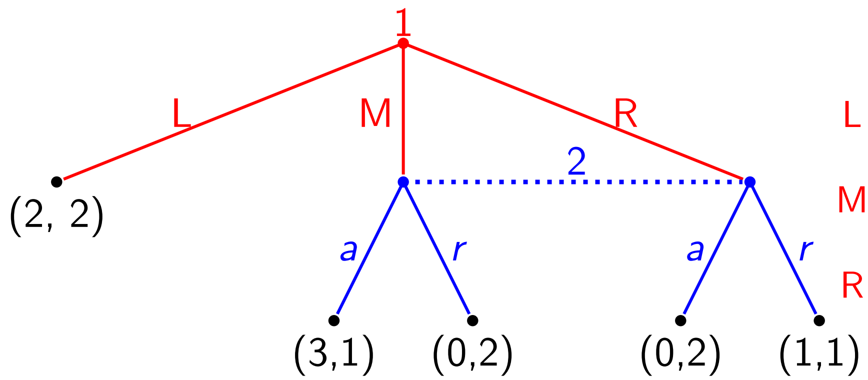
Suppose we want to generalize the idea of subgame perfect equilibrium. Consider the equilibrium (L, r). Is it subgame perfect?

Beliefs

- A belief μ is a function that assigns to every information set a probability measure on the set of histories in the information set.
- An assessment in an extensive-form game is a strategy-belief pair (s, μ).
- The assessment (s, μ) is sequentially rational if for every player i and every information set $\mathcal{I}_{i,j} \in \mathcal{I}_i$ we have

$$E[u_i(s_i, s_{-i}|\mathcal{I}_{i,j})] \geq E[u_i(s_i', s_{-i}|\mathcal{I}_{i,j})]$$

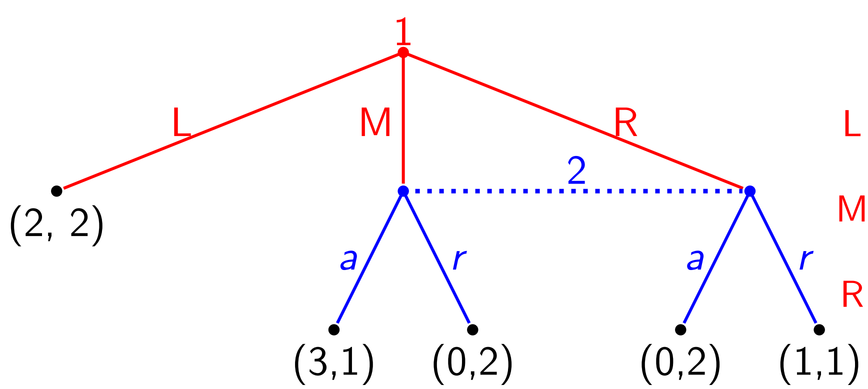
for any $s_i' \neq s_i$.



	a	r
L	2,2	2,2
M	3,1	0,2
R	0,2	1,1

Restrictions to beliefs?

- In lieu of the Nash equilibrium concept, we require that beliefs are derived from equilibrium strategies according to Bays rule (as if players know each others strategies).
- But, what about beliefs for information sets that are off the equilibrium path?

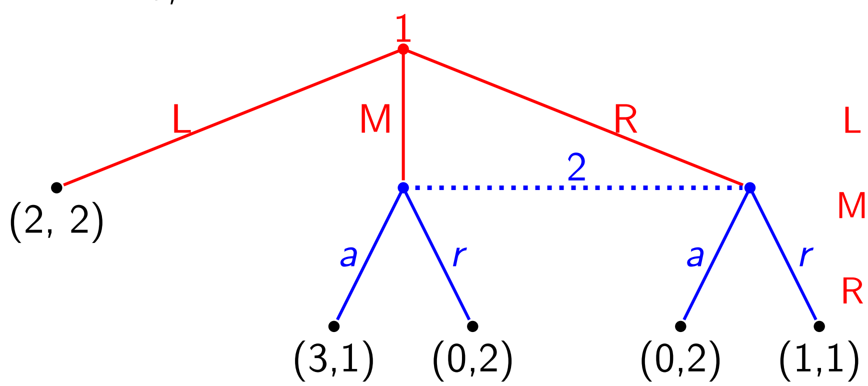


	a	r
L	2,2	2,2
M	3,1	0,2
R	0,2	1,1

Restrictions to Beliefs?

- We want beliefs for information sets that are off the equilibrium path to be reasonable. But what is reasonable?

Consider the NE (L, r) again. Player 2's information set will not be reached at the equilibrium, because player 1 will play L with probability 1. But assume that player 1 plays a completely mixed strategy, playing L, M, and R with probabilities $1-\epsilon$, $\frac{3\epsilon}{4}$, and $\frac{\epsilon}{4}$. Then, the belief on player 2's information set is well defined. Now, if $\epsilon \longrightarrow 0$, it's still well defined.



	a	r
L	2,2	2,2
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Consistent Assessment

- An assessment (s, μ) is consistent if there is a sequence $((s^n, \mu^n))_{n=1}^{\infty}$ of assessments that converges to (s, μ) and has the properties that each strategy profile s^n is completely mixed and that each belief system μ^n is derived from s^n using Bayes rule.

Sequential Equilibrium

- An assessment (s, μ) is a sequential equilibrium of a finite extensive-form game with perfect recall if it is sequentially rational and consistent.
- **Thm**: Every finite extensive-form game with perfect recall has a sequential equilibrium.
- A sequential equilibrium is a Nash equilibrium.
- With perfect information, a subgame perfect equilibrium is a sequential equilibrium.

Bayesian Games

Yiling Chen

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So far

Up to this point, we have assumed that players know all relevant information about each other. Such games are known as games with complete information.

Games with Incomplete Information

- Bayesian Games = Games with Incomplete Information
- Incomplete Information: Players have private information about something relevant to his decision making.
  - Incomplete information introduces uncertainty about the game being played.
- Imperfect Information: Players do not perfectly observe the actions of other players or forget their own actions.

We will see that Bayesian games can be represented as extensive-form games with imperfect information.

Example 4: A Modified Prisoner's Dilemma Game

With probability λ, player 2 has the normal preferences as before (type I), while with probability $(1-\lambda)$, player 2 hates to rat on his accomplice and pays a psychic penalty equal to 6 years in prison for confessing (type II).

λ

	C	D
C	5, 5	0, 8
D	8, 0	1, 1

Type I

$1-\lambda$

	C	D
C	5, 5	0, 2
D	8, 0	1, -5

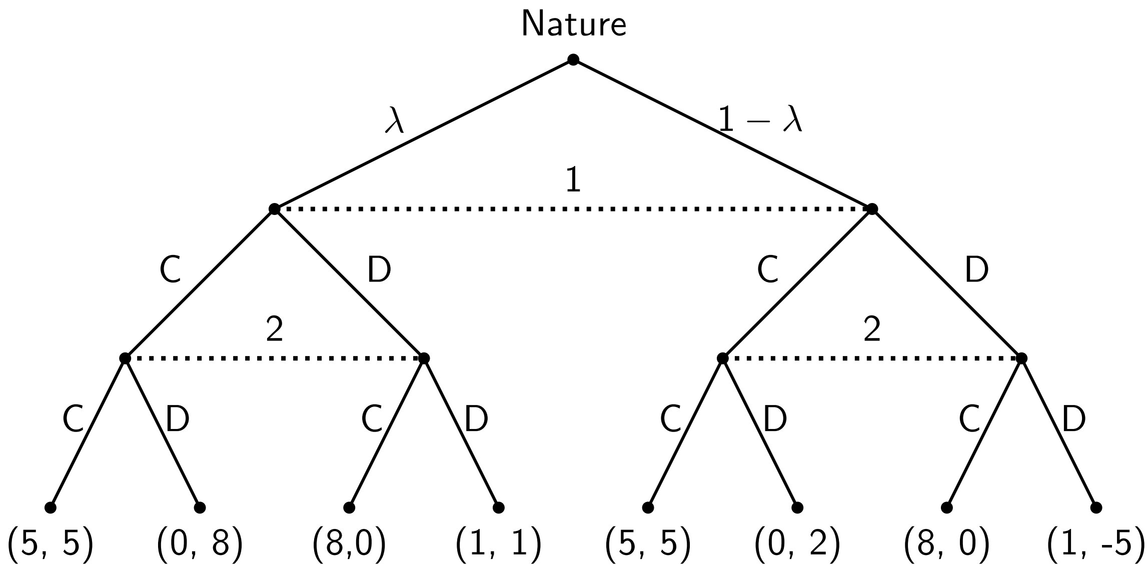
Type II

Simultaneous-Move Bayesian Games

- A simultaneous-move Bayesian game is (N, A, Θ, F, u)
  - $N = \{1, ..., n\}$ is the set of players
  - $A = \{A_1, A_2, ..., A_n\}$ is the set of actions
    $A_i = \{\text{Cooperation, Defection}\}$.
  - $\Theta = \{\Theta_1, \Theta_2, ..., \Theta_n\}$ is the set of types. $\theta_i \in \Theta_i$ is a realization of types for player i.
    $\Theta_2 = \{\text{I, II}\}$.
  - $F : \Theta \to [0, 1]$ is a joint probability distribution, according to which types of players are drawn
    $\text{p}(\theta_2 = \text{type I}) = \lambda$
  - $u = \{u_1, u_2, ..., u_n\}$ where $u_i : A \times \Theta \to \mathcal{R}$ is the utility function of player i
- Two assumptions
  - All possible games have the same number of agents and the same action spaces for each agent
  - Agents have common prior. The different beliefs of agents are posteriors.

Imperfect-Information Extensive-Form Representation of Bayesian Games

- Add a player Nature who has a unique strategy of randomizing in a commonly known way.



Strategies in Bayesian Games

- A pure strategy s_i: $\Theta_i \rightarrow A_i$ of player i is a mapping from every type player i could have to the action he would play if he had that type. Denote the set of pure strategies of player i as S_i.
  $S_1 = \{\{\text{C}\}, \{\text{D}\}\}$
  $S_2 = \{\{\text{C if type I, C if type II}\}, \{\text{C if type I, D if type II}\}, \{\text{D if type I, C if type II}\}, \{\text{D if type I, D if type II}\}\}$
- A mixed strategy $\sigma_i : S_i \rightarrow [0, 1]$ of player i is a distribution over his pure strategies.

Best Response and Bayesian Nash Equilibrium

We use pure strategies to illustrate the concepts. But they hold the same for mixed strategies.

- Player i's ex ante expected utility is

$$E_\theta[u_i(s(\theta),\theta)] = \sum_{\theta_i \in \Theta_i} \mathrm{p}(\theta_i) E_{\theta_{-i}}[u_i(s(\theta),\theta)|\theta_i]$$

- Player i's best responses to $s_{-i}(\theta_{-i})$ is

$$\begin{aligned} \mathrm{BR}_i &= \arg\max_{s_i(\theta_i)\in S_i} E_\theta[u_i(s_i(\theta_i), s_{-i}(\theta_{-i}),\theta)] \\ &= \sum_{\theta_i \in \Theta_i} \mathrm{p}(\theta_i) \left(\arg\max_{s_i(\theta_i)\in S_i} E_{\theta_{-i}}[u_i(s_i(\theta_i), s_{-i}(\theta_{-i}),\theta)|\theta_i] \right) \end{aligned}$$

- A strategy profile $s_i(\theta_i)$ is a Bayesian Nash Equilibrium iif $\forall i \; s_i(\theta_i) \in \mathrm{BR}_i$.

Bayesian Nash Equilibrium: Example 4

- Playing D is a dominant strategy for type I player 2; playing C is a dominant strategy for type II player 2.
- Player 1's expected utility by playing C is $\lambda \times 0 + (1-\lambda) \times 5 = 5 - 5\lambda$.
- Player 1's expected utility by playing D is $\lambda \times 1 + (1-\lambda) \times 8 = 8 - 7\lambda > 5 - 5\lambda$.
- (D, (D if type I, C if type II)) is a BNE of the game.

λ

	C	D
C	5, 5	0, 8
D	8, 0	1, 1

Type I

$1-\lambda$

	C	D
C	5, 5	0, 2
D	8, 0	1, -5

Type II

Example 5: An Exchange Game

- Each of two players receives a ticket t on which there is a number in [0,1].
- The number on a player's ticket is the size of a prize that he may receive.
- The two prizes are identically and independently distributed according to a uniform distribution.
- Each player is asked independently and simultaneously whether he wants to exchange his prize for the other player's prize.
- If both players agree then the prizes are exchanged; otherwise each player receives his own prize.

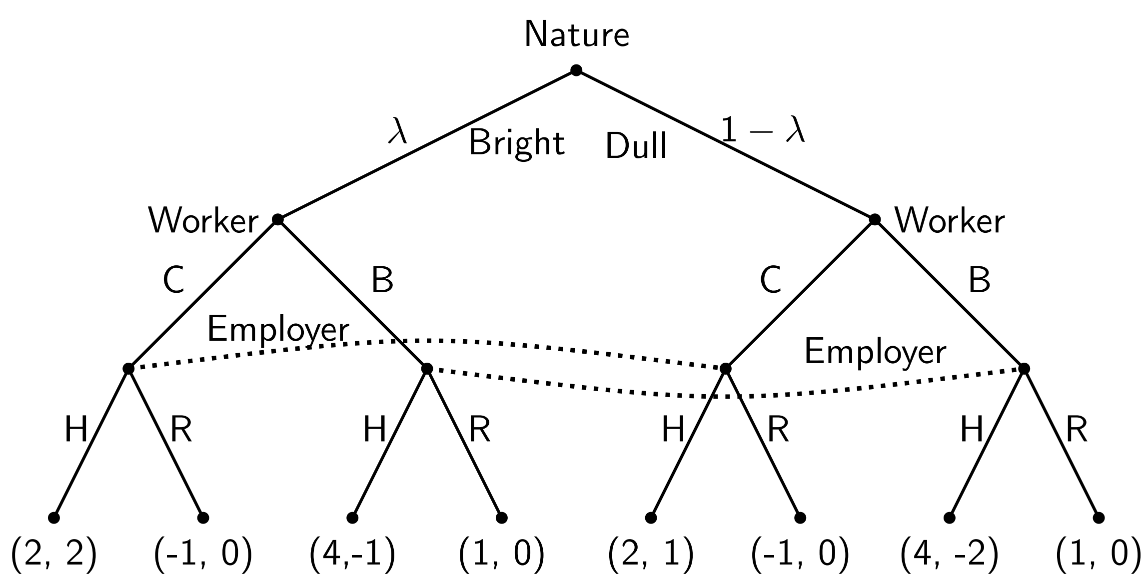
A Bayesian Nash Equilibrium for Example 5

- Strategies of player 1 can be describe as "Exchange if $t_1 \leq k$"
- Given player 1 plays such a strategy, what is the best response of player 2?
  - If $t_2 \geq k$, no exchange
  - If $t_2 < k$, exchange when $t_2 \leq k/2$
- Since players are symmetric, player 1's best response is of the same form.
- Hence, at a Bayesian Nash equilibrium, both players are willing to exchange only when $t_i = 0$.

Signaling (Sender-Receiver Games)

- There are two types of workers, bright and dull.
- Before entering the job market a worker can choose to get an education (i.e. go to college), or enjoy life (i.e. go to beach).
- The employer can observe the educational level of the worker but not his type.
- The employer can hire or reject the worker.

Example 6: Signaling

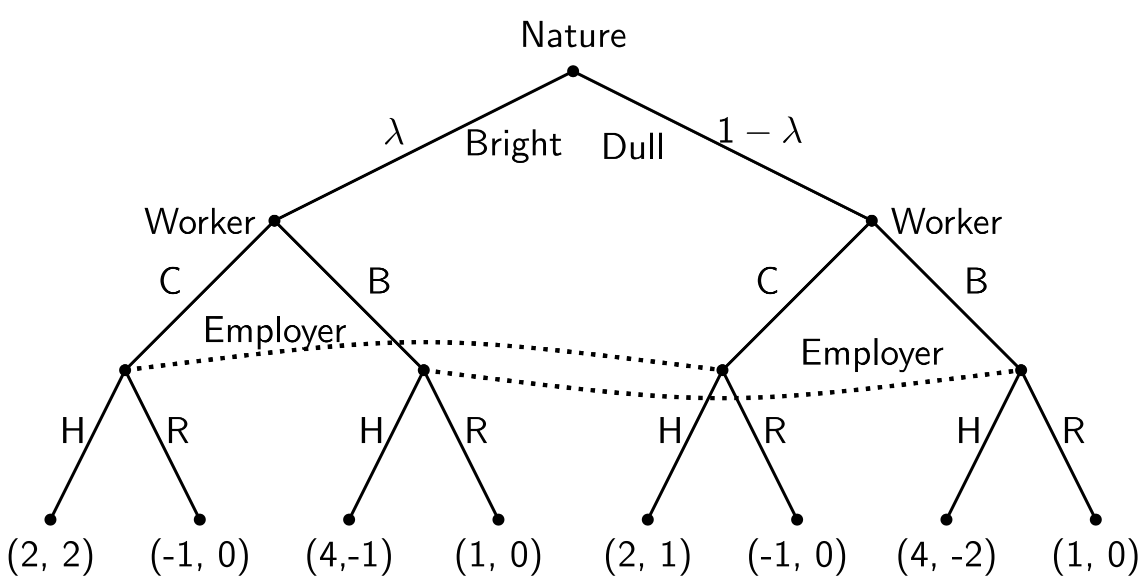


Bayesian Extensive Games with Observable Actions

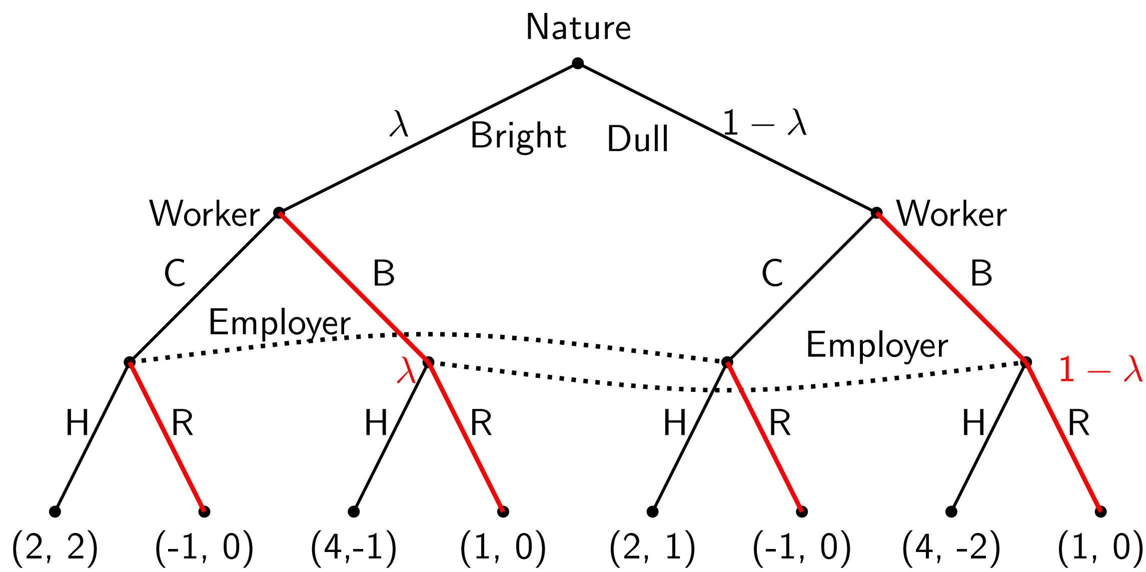
- A Bayesian extensive game with observable actions is (N, H, P, Θ, p, u)
  - (N, H, P) is the same as those in an extensive-form game with perfect information
  - $\Theta = \{\Theta_1, \Theta_2, ..., \Theta_n\}$ is the set of types. $\theta_i \in \Theta_i$ is a realization of types for player i.
    $\Theta_1 = \{$Bright, Dull$\}$.
  - $F : \Theta \to [0, 1]$ is a joint probability distribution, according to which types of players are drawn
    p(θ_1 = Bright) $= \lambda$
  - $u = \{u_1, u_2, ..., u_n\}$ where $u_i : Z \times \Theta \to \mathcal{R}$ is the utility function of player i. $Z \in H$ is the set of terminal histories.

Best Responses for Example 6

- E.g. If the employer always plays H, then the best response for the worker is B.
- But how to define best responses for the employer?
  - Beliefs on information sets
  - Beliefs derived from strategies



A Bayesian Nash Equilibrium of Example 6



$$\text{p(Bright}|\text{Beach)} = \frac{\text{p(Bright)}\sigma(\text{Beach}|\text{Bright})}{\text{p(Bright)}\sigma(\text{Beach}|\text{Bright})+\text{p(Dull)}\sigma(\text{Beach}|\text{Dull})} = \frac{\lambda\cdot 1}{\lambda\cdot 1+(1-\lambda)\cdot 1} = \lambda$$

"Subgame Perfection"

- The previous Bayesian Nash Equilibrium is not "subgame perfect". When the information set College is reached, the employer should choose to hire no matter what belief he has.
- We need to require sequential rationality even for off-equilibrium-path information sets.
- Then, beliefs on off-equilibrium-path information sets matter.

Perfect Bayesian Equilibrium

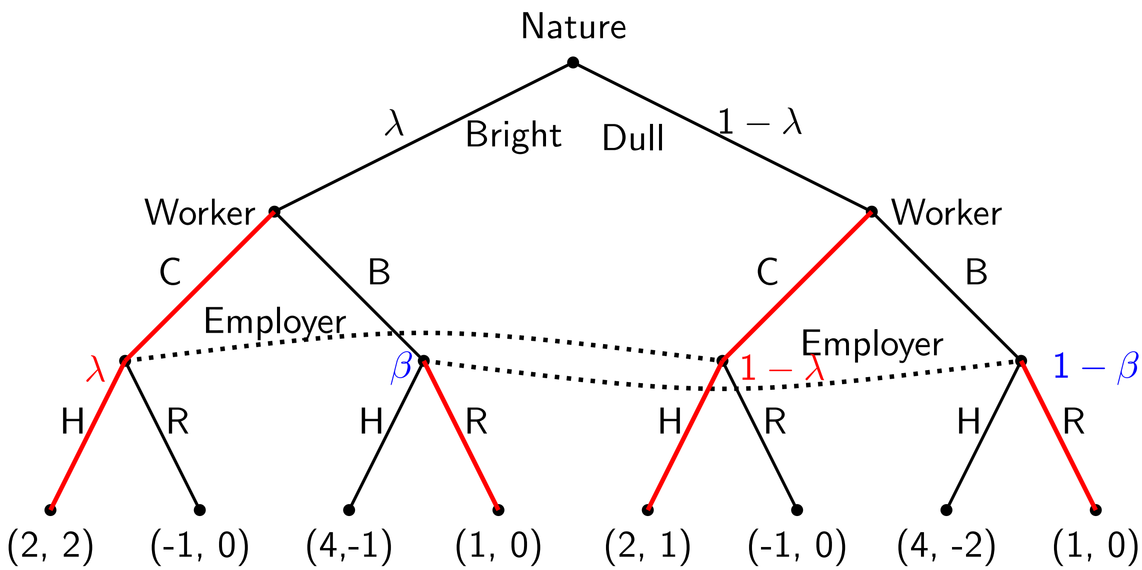
A strategy-belief pair, (σ, μ) is a perfect Bayesian equilibrium if

- (Beliefs) At every information set of player i, the player has beliefs about the node that he is located given that the information set is reached.
- (Sequential Rationality) At any information set of player i, the restriction of (σ, μ) to the continuation game must be a Bayesian Nash equilibrium.
- (On-the-path beliefs) The beliefs for any on-the-equilibrium-path information set must be derived from the strategy profile using Bayes' Rule.
- (Off-the-path beliefs) The beliefs at any off-the-equilibrium-path information set must be determined from the strategy profile according to Bayes Rule whenever possible.

Perfect Bayesian Equilibrium

- Perfect Bayesian equilibrium is a similar concept to sequential equilibrium, both trying to achieve some sort of "subgame perfection".
- Perfect Bayesian equilibrium is defined for all extensive-form games with imperfect information, not just for Bayesian extensive games with observable actions.
- **Thm**: For Bayesian extensive games with observable actions, every sequential equilibrium is a Perfect Bayesian equilibrium.

A Perfect Bayesian Equilibrium of Example 3



$\beta \in [0, 1]$

Summary of Equilibrium Concepts

	On-equ-path strategy σ_{on}	On-equ-path belief μ_{on}	Off-equ-path strategy σ_{off}	Off-equ-path belief μ_{off}
NE	BR	N/A	N/A	N/A
BNE	BR given μ_{on}	Consistent with σ_{on}	N/A	N/A
SPNE	BR	N/A	BR	N/A
PBE	BR given μ_{on}	Consistent with σ_{on}	BR given μ_{off}	Consistent with σ_{off}
SE	BR given μ_{on}	Consistent with σ_{on}	BR given μ_{off}	Consistent with σ_{off}