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Research Article

A Hub Matrix Theory and Applications to Wireless Communications

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This paper considers communications and network systems whose properties are characterized by the gaps of the leading eigenvalues of A^HA for a matrix A. It is shown that a sufficient and necessary condition for a large eigen-gap is that A is a "hub" matrix in the sense that it has dominant columns. Some applications of this hub theory in multiple-input and multiple-output (MIMO) wireless systems are presented.

1. INTRODUCTION

There are many communications and network systems whose properties are characterized by the eigenstructure of a matrix of the form A^HA, also known as the Gram matrix of $\mathbf{A}$, where $\mathbf{A}$ is a matrix with real or complex entries. For example, for a communications system, $\mathbf{A}$ could be a channel matrix, usually denoted $\mathbf{H}$. The capacity of such system is related to the eigenvalues of H^HH [1]. In the area of web page ranking, with entries of $\mathbf{A}$ representing hyperlinks, Kleinberg [2] shows that eigenvectors corresponding to the largest eigenvalues of A^TA give the rankings of the most useful (authority) or popular (hub) web pages. Using a reputation system that parallels Kleinberg's work, Kung and Wu [3] developed an eigenvector-based peer-to-peer (P2P) network user reputation ranking in order to provide services to P2P users based on past contributions (reputation) to avoid "freeloaders." Furthermore, the rate of convergence in the iterative computation of reputations is determined by the gap of the leading two eigenvalues of A^HA.

The recognition that the eigenstructure of A^HA determines the properties of these communications and network systems motivates the work of this paper. We will develop a theoretical framework, called a hub matrix theory, which allows us to predict the eigenstructure of A^HA by examining $\mathbf{A}$ directly. We will prove sufficient and necessary conditions for the existence of a large gap between the largest and the second largest eigenvalues of A^HA. Finally, we apply the "hub" theory and our mathematical results to multiple-input and multiple-output (MIMO) wireless systems.

2. HUB MATRIX THEORY

It is instructive to conduct a thought experiment on a computation process before we introduce our hub matrix theory. The process iteratively computes the values for a set of variables, which for example could be beamforming weights in a beamforming communication system. Figure 1 depicts an example of this process: variable X uses and contributes to variables U_2 and U_4, variable Y uses and contributes to variables U_3 and U_5, and variable Z uses and contributes to all variables $U_1, \dots, U_6$. We say variable Z is a "hub" in the sense that variables involved in Z's computation constitute a superset of those involved in the computation of any other variable. The dominance is illustrated graphically in Figure 1.

We can describe the computation process in matrix notation. Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

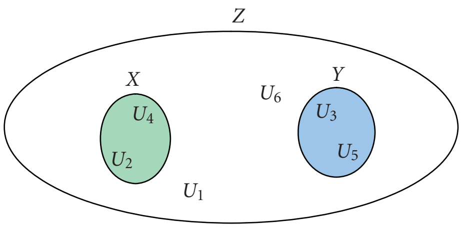

FIGURE 1: Graphical representation of hub concept.

This process performs two steps alternatively (cf. Figure 1).

(1) X, Y, and Z contribute to variables in their respective regions.
(2) X, Y, and Z compute their values using variables in their respective regions.

The first step (1) is $(U_1, U_2, \ldots, U_6)^T \leftarrow A^*(X, Y, Z)^T$ and next step (2) is $(X, Y, Z)^T \leftarrow A^{T*}(U_1, U_2, \ldots, U_6)^T$. Thus, the computational process performs the iteration $(X, Y, Z)^T \leftarrow S^*(X, Y, Z)^T$, where S is defined as follows:

$$S = A^T A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 6 \end{pmatrix}. \quad (2)$$

Note that an arrowhead matrix S, as defined below, has emerged. Furthermore, note that matrix A exhibits the hub property of Z in Figure 1 in view of the fact that the last column of A consists of all 1's, whereas other columns consist of only a few 1's.

Definition 1 (arrowhead matrix). Let $S \in \mathbf{C}^{m\times m}$ be a given Hermitian matrix. S is called an *arrowhead matrix* if

$$S = \begin{pmatrix} D & c \\ c^H & b \end{pmatrix}, \quad (3)$$

where $D = \operatorname{diag}(d^{(1)}, \ldots, d^{(m-1)}) \in \mathbf{R}^{(m-1)\times(m-1)}$ is a real diagonal matrix, $c = (c^{(1)}, \ldots, c^{(m-1)}) \in \mathbf{C}^{m-1}$ is a complex vector, and $b \in \mathbf{R}$ is a real number.

The eigenvalues of an arbitrary square matrix are invariant under similarity transformations. Therefore, we can with no loss of generality arrange the diagonal elements of D to be ordered so that $d^{(i)} \le d^{(i+1)}$ for $i = 1, \ldots, m-2$. For details concerning arrowhead matrices, see for example [4].

Definition 2 (hub matrix). A matrix $A \in \mathbf{C}^{n\times m}$ is called a *candidate-hub matrix*, if $m-1$ of its columns are orthogonal to each other with respect to the Euclidean inner product. If in addition the remaining column has its Euclidean norm greater than or equal to that of any other column, then the matrix A is called a *hub matrix* and this remaining column is called the *hub column*. We are normally interested in hub matrices where the hub column has much large magnitude than other columns. (As we show later in Theorems 4 and 10 that in this case the corresponding arrowhead matrices will have large eigengaps).

In this paper, we study the eigenvalues of $S = A^H A$, where A is a hub matrix. Since the eigenvalues of S are invariant under similarity transformations of S, we can permute the columns of the hub matrix A so that its last column is the hub column without loss of generality. For the rest of this paper, we will denote the columns of a hub matrix A by $a_1, \ldots, a_m$, and assume that columns $a_1, \ldots, a_{m-1}$ are orthogonal to each other, that is, $a_i^H a_j = 0$ for $i \ne j$ and $i, j = 1, \ldots, m-1$, and column a_m is the hub column. The matrix A introduced in the context of the graphical model from Figure 1 is such a hub matrix.

In Section 4, we will relax the orthogonality condition of a hub matrix, by introducing the notion of hub and arrowhead dominant matrices.

Theorem 1. *Let* $\mathbf{A} \in \mathbf{C}^{n\times m}$ *and let* $S \in \mathbf{C}^{m\times m}$ *be the Gram matrix of* $\mathbf{A}$ *that is,* $S = A^H A$. $\mathbf{S}$ *is an arrowhead matrix if and only if* $\mathbf{A}$ *is a candidate-hub matrix.*

Proof. Suppose A is a candidate-hub matrix. Since $S = A^H A$, the entries of S are $s^{(i,j)} = a_i^H a_j$ for $i, j = 1, \ldots, m$. By Definition 2 of a candidate-hub matrix, the nonhub columns of A are orthogonal, that is, $a_i^H a_j = 0$ for $i \ne j$ and $i, j = 1, \ldots, m-1$. Since S is Hermitian, the transpose of the last column is the complex conjugate of the last row and the diagonal elements of S are real numbers. Therefore, $S = A^H A$ is an arrowhead matrix by Definition 1.

Suppose $S = A^H A$ is an arrowhead matrix. Note that the components of the S matrix of Definition 1 can be represented in terms of the inner products of columns of A, that is, $b = a_m^H a_m$, $d^{(i)} = a_i^H a_i$, $c^{(i)} = a_i^H a_m$ for $i = 1, \ldots, m-1$. Since S is an arrowhead matrix, all other off-diagonal entries of S, $s^{(i,j)} = a_i^H a_j$ for $i \ne j$ and $i, j = 1, \ldots, m-1$, are zero. Thus, $a_i^H a_j = 0$ if $i \ne j$ and $i, j = 1, \ldots, m-1$. So, A is a candidate-hub matrix by Definition 2. □

Before proving our main result in Theorem 4, we first restate some well-known results which will be needed for the proof.

Theorem 2 (interlacing eigenvalues theorem for bordered matrices). *Let* $U \in \mathbf{C}^{(m-1)\times(m-1)}$ *be a given Hermitian matrix, let* $y \in \mathbf{C}^{(m-1)}$ *be a given vector, and let* $a \in \mathbf{R}$ *be a given real number. Let* $V \in \mathbf{C}^{m\times m}$ *be the Hermitian matrix obtained by bordering* U *with* y *and* a *as follows:*

$$V = \begin{pmatrix} U & y \\ y^H & a \end{pmatrix}. \quad (4)$$

Let the eigenvalues of V *and* U *be denoted by* $\{\lambda_i\}$ *and* $\{\mu_i\}$*, respectively, and assume that they have been arranged in increasing order, that is,* $\lambda_1 \le \cdots \le \lambda_m$ *and* $\mu_1 \le \cdots \le \mu_{m-1}$. *Then*

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \cdots \le \lambda_{m-1} \le \mu_{m-1} \le \lambda_m. \quad (5)$$

Proof. See [5, page 189]. □

Definition 3 (majorizing vectors). Let $\boldsymbol{\alpha} \in \mathbf{R}^m$ and $\boldsymbol{\beta} \in \mathbf{R}^m$ be given vectors. If we arrange the entries of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in increasing order, that is, $\alpha^{(1)} \leq \cdots \leq \alpha^{(m)}$ and $\beta^{(1)} \leq \cdots \leq \beta^{(m)}$, then vector $\boldsymbol{\beta}$ is said to majorize vector $\boldsymbol{\alpha}$ if

$$\sum_{i=1}^{k} \beta^{(i)} \geq \sum_{i=1}^{k} \alpha^{(i)} \quad \text{for } k = 1, \dots, m \tag{6}$$

with equality for $k = m$.

For details concerning majorizing vectors, see [5, pages 192–198]. The following theorem provides an important property expressed in terms of vector majorizing.

Theorem 3 (Schur-Horn theorem). *Let $V \in \mathbf{C}^{m\times m}$ be Hermitian. The vector of diagonal entries of V majorizes the vector of eigenvalues of V.*

Proof. See [5, page 193]. □

Definition 4 (hub-gap). Let $A \in \mathbf{C}^{n\times m}$ be a matrix with its columns denoted by $a_1, \dots, a_m$ with $0 < \|a_1\|_2^2 \leq \cdots \leq \|a_m\|_2^2$. For $i = 1, \dots, m-1$, the ith hub-gap of A is defined to be

$$\text{HubGap}_i(A) = \frac{\|a_{m-(i-1)}\|_2^2}{\|a_{m-i}\|_2^2}. \tag{7}$$

Definition 5 (eigengap). Let $S \in \mathbf{C}^{m\times m}$ be a Hermitian matrix with its real eigenvalues denoted by $\lambda_1, \dots, \lambda_m$ with $\lambda_1 \leq \cdots \leq \lambda_m$. For $i = 1, \dots, m-1$, the ith eigengap of S is defined to be

$$\text{EigenGap}_i(S) = \frac{\lambda_{m-(i-1)}}{\lambda_{m-i}}. \tag{8}$$

Theorem 4. *Let $A \in \mathbf{C}^{n\times m}$ be a hub matrix with its columns denoted by $a_1, \dots, a_m$ and $0 < \|a_1\|_2^2 \leq \cdots \leq \|a_m\|_2^2$. Let $S = A^H A \in \mathbf{C}^{m\times m}$ be the corresponding arrowhead matrix with its eigenvalues denoted by $\lambda_1, \dots, \lambda_m$ with $0 \leq \lambda_1 \leq \cdots \leq \lambda_m$. Then*

$$\begin{aligned} \text{HubGap}_1(A) &\leq \text{EigenGap}_1(S) \\ &\leq \left(\text{HubGap}_1(A) + 1\right) \text{HubGap}_2(A). \end{aligned} \tag{9}$$

Proof. Let T be the matrix formed from S by deleting its last row and column. This means that T is a diagonal matrix with diagonal elements $\|a_i\|_2^2$ for $i = 1, \dots, m-1$. By Theorem 2, the eigenvalues of S interlace those of T, that is, $\lambda_1 \leq \|a_1\|_2^2 \leq \cdots \leq \lambda_{m-1} \leq \|a_{m-1}\|_2^2 \leq \lambda_m$. Thus, λ_{m-1} is a lower bound for $\|a_{m-1}\|_2^2$. By Theorem 3, the vector of diagonal values of S majorizes the vector of eigenvalues of S, that is, $\sum_{i=1}^{k} d^{(i)} \geq \sum_{i=1}^{k} \lambda_i$ for $k = 1, \dots, m-1$ and $\sum_{i=1}^{m-1} d^{(i)} + b = \sum_{i=1}^{m} \lambda_m$. So, $b \leq \lambda_m$. Since $b = \|a_m\|_2^2$, λ_m is an upper bound for $\|a_m\|_2^2$. Hence, $\|a_m\|_2^2/\|a_{m-1}\|_2^2 \leq \lambda_m/\lambda_{m-1}$ or $\text{HubGap}_1(A) \leq \text{EigenGap}_1(S)$.

Again, by using Theorems 2 and 3, we have $\sum_{i=1}^{m-1} d^{(i)} + b = \sum_{i=1}^{m} \lambda_m$ and $\lambda_1 \leq d^{(1)} \leq \lambda_2 \leq d^{(2)} \leq \lambda_3 \leq \cdots \leq d^{(m-2)} \leq \lambda_{m-1} \leq d^{(m-1)} \leq \lambda_m$, and, as such,

$$\begin{aligned} &\left(d^{(1)} + \cdots + d^{(m-2)}\right) + d^{(m-1)} + b \\ &\quad = \lambda_1 + \left(\lambda_2 + \cdots + \lambda_{m-1}\right) + \lambda_m \\ &\quad \geq \lambda_1 + \left(d^{(1)} + \cdots + d^{(m-2)}\right) + \lambda_m. \end{aligned} \tag{10}$$

This result implies that $d^{(m-1)} + b \geq \lambda_1 + \lambda_m \geq \lambda_m$. By noting that $d^{(m-2)} \leq \lambda_{m-1}$, we have

$$\begin{aligned} \text{EigenGap}_1(S) = \frac{\lambda_m}{\lambda_{m-1}} \leq \frac{d^{(m-1)} + b}{d^{(m-2)}} &= \frac{\|a_{m-1}\|_2^2 + \|a_m\|_2^2}{\|a_{m-2}\|_2^2} \\ &= \frac{\|a_{m-1}\|_2^2}{\|a_{m-2}\|_2^2} + \frac{\|a_m\|_2^2}{\|a_{m-1}\|_2^2} \cdot \frac{\|a_{m-1}\|_2^2}{\|a_{m-2}\|_2^2} \\ &= \left(\text{HubGap}_1(A) + 1\right) \cdot \text{HubGap}_2(A). \end{aligned} \tag{11}$$

□

By Theorem 4, we have the following result, where notation "≫" means "much larger than."

Corollary 1. *Let $A \in \mathbf{C}^{n\times m}$ be a matrix with its columns $a_1, \dots, a_m$ satisfying $0 < \|a_1\|_2^2 \leq \cdots \leq \|a_{m-1}\|_2^2 \leq \|a_m\|_2^2$. Let $S = A^H A \in \mathbf{C}^{m\times m}$ with its eigenvalues $\lambda_1, \cdots, \lambda_m$ satisfying $0 \leq \lambda_1 \leq \cdots \leq \lambda_m$. The following holds*

(1) *if A is a hub matrix with $\|a_m\|_2 \gg \|a_{m-1}\|_2$, then S is an arrowhead matrix with $\lambda_m \gg \lambda_{m-1}$; and*
(2) *if S is an arrowhead matrix with $\lambda_m \gg \lambda_{m-1}$, then A is a hub matrix with $\|a_m\|_2 \gg \|a_{m-1}\|_2$ or $\|a_{m-1}\|_2 \gg \|a_{m-2}\|_2$ or both.*

3. MIMO COMMUNICATIONS APPLICATION

A multiple-input multiple-output (MIMO) system with M_t transmit antennas and M_r receive antennas is depicted in Figure 2 [6, 7]. Assume the MIMO channel is modeled by the $M_r \times M_t$ channel propagation matrix $H = (h_{ij})$. The input-output relationship, given a transmitted symbol s, for this system is given by

$$x = s z^H H w + z^H n. \tag{12}$$

The vectors w and z in the equation are called the beamforming and combining vectors, respectively, which will be chosen to maximize the signal-to-noise ratio (SNR). We will model the noise vector n as having entries, which are independent and identically distributed (i.i.d.) random variables of complex Gaussian distribution $CN(0,1)$. Without loss of generality, assume the average power of transmit signal equals one, that is, $E|s|^2 = 1$. For the beamforming system described here, the signal to noise ratio, γ, after combining at the receiver is given by

$$\gamma = \frac{\left|z^H H w\right|^2}{\|z\|_2^2}. \tag{13}$$

Without loss of generality, assume $\|z\|_2 = 1$. With this assumption, the SNR becomes

$$\gamma = \left|z^H H w\right|^2. \tag{14}$$

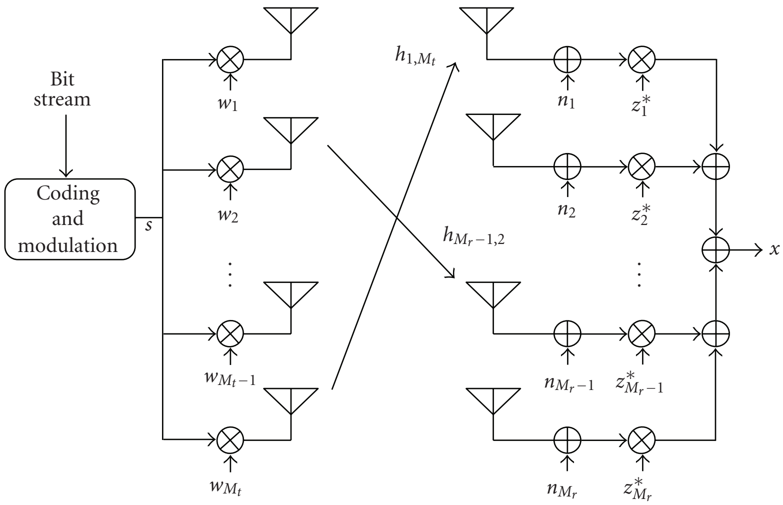

FIGURE 2: MIMO block diagram (see [6, datapath portion of Figure 1]).

3.1. Maximum ratio combining

A receiver where z maximizes γ for a given w is known as a maximum ratio combining (MRC) receiver in the literature. By the Cauchy-Bunyakovskii-Schwartz inequality (see, e.g., [8, page 272]), we have

$$|z^H Hw|^2 \leq \|z\|_2^2 \|Hw\|_2^2. \tag{15}$$

Since we already assume $\|z\|_2 = 1$,

$$|z^H Hw|^2 \leq \|Hw\|_2^2. \tag{16}$$

Moreover, since in MRC we desire to maximize the SNR, we must choose z to be

$$z^{\text{MRC}} = \frac{Hw}{\|Hw\|_2}, \tag{17}$$

which implies that the SNR for MRC is

$$\gamma^{\text{MRC}} = \|Hw\|_2^2. \tag{18}$$

3.2. Selection diversity transmission, generalized subset selection, and combined SDT/MRC and GSS/MRC

For a selection diversity transmission (SDT) [9] system, only the antenna that yields the largest SNR is selected for transmission at any instant of time. This means

$$w = [\delta_{1,f(1)}, \ldots, \delta_{M_t,f(1)}]^T, \tag{19}$$

where the Kronecker impulse $\delta_{i,j}$ is defined as $\delta_{i,j} = 1$ if $i = j$, and $\delta_{i,j} = 0$ if $i \neq j$, and $f(1)$ represents the value of the index x that maximizes $\sum_i |h_{i,x}|^2$. Thus, the SNR for the combined SDT/MRC communications system is

$$\gamma^{\text{SDT/MRC}} = ||h_{f(1)}||_2^2. \tag{20}$$

By definition, a generalized subset selection (GSS) [10] system powers those k transmitters which yield the top k SNR values at the receiver for some $k > 1$. That is, if $f(1), f(2), \ldots, f(k)$ stand for the indices of these transmitters, then $w_{f(i)} = 1/\sqrt{k}$ for $i = 1, \ldots, k$, and all other entries of w are zero. It follows that, for the combined GSS/MRC communications system, the SNR gain is given by

$$\gamma^{\text{GSS/MRC}} = \frac{1}{k}\left\|\sum_{i=1}^{k} h_{f(i)}\right\|_2^2. \tag{21}$$

In the limiting case when $k = M_t$, GSS becomes equal gain transmission (EGT) [6, 7], which requires all M_t transmitters to be equally powered, that is, $w_{f(i)} = 1/\sqrt{M_t}$ for $i = 1, \ldots, M_t$. Then, for the combined EGT/MRC communications system, the SNR gain takes the expression

$$\gamma^{\text{EGT/MRC}} = \frac{1}{M_t}\left\|\sum_{i=1}^{M_t} h_{f(i)}\right\|_2^2. \tag{22}$$

3.3. Maximum ratio transmission and combined MRT/MRC

Suppose there are no constraints placed on the form of the vector w. Let us reexamine the expression of SNR gain γ^{MRC}. Note

$$\gamma^{\text{MRC}} = \|Hw\|_2^2 = (Hw)^H(Hw) = w^H(H^H Hw). \tag{23}$$

With the assumption that $\|w\|_2 = 1$, the above equation is maximized under maximum ratio transmission (MRT) [9] (see, e.g., [5, page 295]), that is, when

$$w = w_m, \tag{24}$$

where w_m is the normalized eigenvector corresponding to the largest eigenvalues λ_m of $H^H H$. Thus, for an MRT/MRC system, we have

$$\gamma^{\text{MRT/MRC}} = \lambda_m. \tag{25}$$

3.4. Performance comparison between SDT/MRC and MRT/MRC

Theorem 5. *Let $H \in \mathbf{C}^{n\times m}$ be a hub matrix with its columns denoted by $h_1,\dots,h_m$ and $0 < \|h_1\|_2^2 \le \cdots \le \|h_{m-1}\|_2^2 \le \|h_m\|_2^2$. Let $\gamma^{\text{SDT/MRC}}$ and $\gamma^{\text{MRT/MRC}}$ be the SNR gains for SDT/MRC and MRT/MRC, respectively. Then*

$$\frac{\text{HubGap}_1(H)}{\text{HubGap}_1(H)+1} \le \frac{\gamma^{\text{SDT/MRC}}}{\gamma^{\text{MRT/MRC}}} \le 1. \tag{26}$$

Proof. We note that the A matrix in hub matrix theory of Section 2 corresponds to the H matrix here, and the a_i column of A corresponds to the h_i column of H for $i = 1,\dots,m$. From the proof of Theorem 4, we note $b = \|a_m\|_2^2 \le \lambda_m$ or $\|h_m\|_2^2 \le \lambda_m$. It follows that

$$\frac{\gamma^{\text{SDT/MRC}}}{\gamma^{\text{MRT/MRC}}} \le 1. \tag{27}$$

To derive a lower bound for $\gamma^{\text{SDT/MRC}}/\gamma^{\text{MRT/MRC}}$, we note from the proof of Theorem 4 that $\lambda_m \le d^{(m-1)} + b$. This means that

$$\gamma^{\text{MRT/MRC}} \le ||a_{m-1}||_2^2 + ||a_m||_2^2 = ||h_{m-1}||_2^2 + ||h_m||_2^2. \tag{28}$$

Thus

$$\frac{\gamma^{\text{SDT/MRC}}}{\gamma^{\text{MRT/MRC}}} \ge \frac{||h_m||_2^2}{||h_{m-1}||_2^2 + ||h_m||_2^2} = \frac{\text{HubGap}_1(H)}{\text{HubGap}_1(H)+1}. \tag{29}$$

□

The inequality $\gamma^{\text{SDT/MRC}}/\gamma^{\text{MRT/MRC}} \le 1$ in Theorem 5 reflects the fact that in the SDT/MRC system, w is chosen to be a particular unit vector rather than an optimal choice. The other inequality of Theorem 5, $\text{HubGap}_1(H)/(\text{HubGap}_1(H) + 1) \le \gamma^{\text{SDT/MRC}}/\gamma^{\text{MRT/MRC}}$, implies that the SNR for SDT/MRC approaches that for MRT/MRC when H is a hub matrix with a dominant hub column. More precisely, we have the following result.

Corollary 2. *Let $H \in \mathbf{C}^{n\times m}$ be a hub matrix with its columns denoted by $h_1,\dots,h_m$ and $0 < \|h_1\|_2^2 \le \cdots \le \|h_m\|_2^2$. Let $\gamma^{\text{SDT/MRC}}$ and $\gamma^{\text{MRT/MRC}}$ be the SNR for SDT/MRC and MRT/MRC, respectively. Then, as $\text{HubGap}_1(H)$ increases, $\gamma^{\text{MRT/MRC}}/\gamma^{\text{SDT/MRC}}$ approaches one at a rate of at least $\text{HubGap}_1(H)/(\text{HubGap}_1(H)+1)$.*

3.5. GSS-MRT/MRC and performance comparison with MRT/MRC

Using an analysis similar to the one above, we can derive performance bounds for a recently discovered communication system that incorporates antenna selection with MRT on the transmission side while applying MRC on the receiver side [11, 12]. This approach will be called GSS-MRT/MRC here. Given a GSS scheme that powers those k transmitters which yield the top k highest SNR values, a GSS-MRT/MRC system is defined to be an MRT/MRC system applied to these k transmitters. Let $f(1), f(2),\dots, f(k)$ be the indices of these k transmitters, and $\tilde{H}$ the matrix formed by columns $h_{f(i)}$ of H for $i = 1,\dots,k$. It is easy to see that the SNR for GSS-MRT/MRC is

$$\gamma^{\text{GSS-MRT/MRC}} = \tilde{\lambda}_m, \tag{30}$$

where $\tilde{\lambda}_m$ is the largest eigenvalue of $\tilde{H}^H\tilde{H}$.

Theorem 6. *Let $H \in \mathbf{C}^{n\times m}$ be a hub matrix with its columns denoted by $h_1,\dots,h_m$ and $0 < \|h_1\|_2^2 \le \cdots \le \|h_{m-1}\|_2^2 \le \|h_m\|_2^2$. Let $\gamma^{\text{GSS}-\text{MRT/MRC}}$ and $\gamma^{\text{MRT/MRC}}$ be the SNR values for GSS-MRT/MRC and MRT/MRC, respectively. Then*

$$\frac{\text{HubGap}_1(H)}{\text{HubGap}_1(H)+1} \le \frac{\gamma^{\text{GSS}-\text{MRT/MRC}}}{\gamma^{\text{MRT/MRC}}} \le \frac{\text{HubGap}_1(H)+1}{\text{HubGap}_1(H)}. \tag{31}$$

Proof. Since $0 < \|h_1\|_2^2 \le \cdots \le \|h_{m-1}\|_2^2 \le \|h_m\|_2^2$, $\tilde{H}$ consists of the last k columns of H. Moreover, since H is a hub matrix, so is $\tilde{H}$. From the proof of Theorem 4, we note both λ_m and $\tilde{\lambda}_m$ are bounded above by $\|h_{m-1}\|_2^2+\|h_m\|_2^2$ and below by $\|h_m\|_2^2$. It follows that

$$\begin{aligned}\frac{\text{HubGap}_1(H)}{\text{HubGap}_1(H)+1} &= \frac{||h_m||_2^2}{||h_{m-1}||_2^2 + ||h_m||_2^2} \le \frac{\gamma^{\text{GSS}-\text{MRT/MRC}}}{\gamma^{\text{MRT/MRC}}} = \frac{\tilde{\lambda}_m}{\lambda_m} \\ &\le \frac{||h_{m-1}||_2^2 + ||h_m||_2^2}{||h_m||_2^2} = \frac{\text{HubGap}_1(H)+1}{\text{HubGap}_1(H)}.\end{aligned} \tag{32}$$

□

3.6. Diversity selection with partitions, DSP-MRT/MRC, and performance bounds

Suppose that transmitters are partitioned into multiple transmission partitions. We define the diversity selection with partitions (DSP) to be the transmission scheme where in each transmission partition only the transmitter with the largest SNR will be powered. Note that SDT discussed above is a special case of DSP when there is only one partition consisting of all transmitters.

Let k be the number of partitions, and $f(1), f(2),\dots, f(k)$ the indices of the powered transmitters. A DSP-MRT/MRC system is defined to be an MRT/MRC system applied to these k transmitters. Define $\hat{H}$ to be the matrix formed by columns $h_{f(i)}$ of H for $i = 1,\dots,k$. Then the SNR for DSP-MRT/MRC is

$$\gamma^{\text{DSPS-MRT/MRC}} = \hat{\lambda}_m, \tag{33}$$

where $\hat{\lambda}_m$ is the largest eigenvalue of $\hat{H}^H\hat{H}$.

Note that in general the powered transmitters for DSP are not the same as those for GSS. This is because a transmitter that yields the highest SNR among transmitters in one of the k partitions may not be among the transmitters that yield the top k highest SNR values among all transmitters. Nevertheless, when H is a hub matrix with

$0 < \|h_1\|_2^2 \leq \cdots \leq \|h_{m-1}\|_2^2 \leq \|h_m\|_2^2$, we can bound $\widehat{\lambda}_m$ for DSP-MRT/MRC in a manner similar to how we bound $\widetilde{\lambda}_m$ for GSS-MRT/MRC. That is, for DSP-MRT/MRC, $\widehat{\lambda}_m$ is bounded above by $\|h_k\|_2^2 + \|h_m\|_2^2$ and below by $\|h_m\|_2^2$, where h_k is the second largest column of $\widehat{H}$ in magnitude. Note that $\|h_k\|_2^2 \leq \|h_{m-1}\|_2^2$, since the second largest column of $\widehat{H}$ in magnitude cannot be larger that than of H. We have the following result similar to that of Theorem 6.

Theorem 7. *Let $H \in \mathbf{C}^{n\times m}$ be a hub matrix with its columns denoted by $h_1, \dots, h_m$ and $0 < \|h_1\|_2^2 \leq \cdots \leq \|h_{m-1}\|_2^2 \leq \|h_m\|_2^2$. Let $\gamma^{\text{DSP-MRT/MRC}}$ and $\gamma^{\text{MRT/MRC}}$ be the SNR for DSP-MRT/MRC and MRT/MRC, respectively. Then*

$$\frac{\text{HubGap}_1(H)}{\text{HubGap}_1(H)+1} \leq \frac{\gamma^{DSP-\text{MRT/MRC}}}{\gamma^{\text{MRT/MRC}}} \leq \frac{\text{HubGap}_1(H)+1}{\text{HubGap}_1(H)}. \tag{34}$$

Theorems 6 and 7 imply that when $\text{HubGap}_1(H)$ *becomes large, the SNR values of both GSS-MRT/MRC and DSP-MRT/MRC approach that of MRT/MRC.*

4. HUB DOMINANT MATRIX THEORY

We generalize the hub matrix theory presented above to situations when matrix A (or H) exhibits a "near" hub property. In order to relax the definition of orthogonality of a set of vectors, we use the notion of frame.

Definition 6 (frame). A set of distinct vectors $\{f_1, \dots, f_n\}$ is said to be a frame if there exist positive constants ξ and ϑ called frame bounds such that

$$\xi\|f_j\|^2 \leq \sum_{i=1}^{n} |f_i^H f_j| \leq \vartheta\|f_j\|^2 \quad \text{for } j = 1, \dots, n. \tag{35}$$

Note that if $\xi = \vartheta = 1$, then the set of vectors $\{f_1, \dots, f_n\}$ is orthogonal. Here we use frames to bound the nonorthogonality of a collection of vectors, while the usual use for frames is to quantify the redundancy in a representation (see, e.g., [13]).

Definition 7 (hub dominant matrix). A matrix $A \in \mathbf{C}^{n\times m}$ is called a *candidate-hub-dominant matrix* if $m-1$ of its columns form a frame with frame bounds $\xi = 1$ and $\vartheta = 2$, that is, $\|a_j\|^2 \leq \sum_{i=1}^{m-1} |a_i^H a_j| \leq 2\|a_j\|^2$ for $j = 1, \dots, m-1$. If in addition the remaining column has its Euclidean norm greater than or equal to that of any other column, then the matrix A is called a *hub-dominant matrix* and the remaining column is called the *hub column*.

We next generalize the definition of arrowhead matrix to arrowhead dominant matrix, where the matrix D in Definition 1 goes from being a diagonal matrix to a diagonally dominant matrix.

Definition 8 (diagonally dominant matrix). Let $E \in \mathbf{C}^{m\times m}$ be a given Hermitian matrix. E is said to be diagonally dominant if for each row the magnitude of the diagonal entry is greater than or equal to the row sum of magnitudes of all off-diagonal entries, that is,

$$|e^{(i,i)}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{m-1} |e^{(i,j)}| \quad \text{for } i = 1, \dots, m. \tag{36}$$

For more information on diagonally dominant matrices, see for example [5, page 349].

Definition 9 (arrowhead dominant matrix). Let $S \in \mathbf{C}^{m\times m}$ be a given Hermitian matrix. S is called an *arrowhead dominant matrix* if

$$S = \begin{pmatrix} D & c \\ c^H & b \end{pmatrix}, \tag{37}$$

where $D \in \mathbf{C}^{(m-1)\times(m-1)}$ is a diagonally dominant matrix, $c = (c^{(1)}, \dots, c^{(m-1)}) \in \mathbf{C}^{m-1}$ is a complex vector, and $b \in \mathbf{R}$ is a real number.

Similar to Theorem 1, we have the following theorem.

Theorem 8. *Let $A \in \mathbf{C}^{n\times m}$ and let $S \in \mathbf{C}^{m\times m}$ be the Gram matrix of A, that is, $S = A^H A$. S is an arrowhead dominant matrix if and only if A is a candidate-hub-dominant matrix.*

Proof. Suppose A is a candidate-hub-dominant matrix. Since $S = A^H A$, the entries of S can be expressed as $s^{(i,j)} = a_i^H a_j$ for $i, j = 1, \dots, m$. By Definition 7 of a hub-dominant matrix, the nonhub columns of A form a frame with frame bounds $\xi = 1$ and $\vartheta = 2$, that is $\|a_j\|^2 \leq \sum_{i=1}^{m-1} |a_i^H a_j| \leq 2\|a_j\|^2$ for $j = 1, \dots, m-1$. Since $\|a_j\|^2 = |a_j^H a_j|$, it follows that $|a_i^H a_i| \geq \sum_{j=1, j\neq i}^{m-1} |a_i^H a_j|$, $i = 1, \dots, m-1$, which is the diagonal dominance condition on the sub-matrix D of S. Since S is Hermitian, the transpose of the last column is the complex conjugate of the last row and the diagonal elements of S are real numbers. Therefore, $S = A^H A$ is an arrowhead dominant matrix in accordance with Definition 9.

Suppose $S = A^H A$ is an arrowhead dominant matrix. Note that the components of the S matrix of Definition 9 can be represented in terms of the columns of A. Thus $b = a_m^H a_m$ and $c^{(i)} = a_i^H a_m$ for $i = 1, \dots, m-1$. Since $|a_j^H a_j| = \|a_j\|^2$, the diagonal dominance condition, $|a_i^H a_i| \geq \sum_{j=1, j\neq i}^{m-1} |a_i^H a_j|$, $i = 1, \dots, m-1$, implies that $\|a_j\|^2 \leq \sum_{i=1}^{m-1} |a_i^H a_j| \leq 2\|a_j\|^2$ for $j = 1, \dots, m-1$. So, A is a candidate-hub-dominant matrix by Definition 7. □

Before proceeding to our results in Theorem 10, we will first restate a well-known result which will be needed for the proof.

Theorem 9 (monotonicity theorem). *Let $G, H \in \mathbf{C}^{m\times m}$ be Hermitian. Assume H is positive semidefinite and that the eigenvalues of G and $G+H$ are arranged in increasing order, that is, $\lambda_1(G) \leq \cdots \leq \lambda_m(G)$ and $\lambda_1(G+H) \leq \cdots \leq \lambda_m(G+H)$. Then $\lambda_\kappa(G) \leq \lambda_k(G+H)$ for $k = 1, \dots, m$.*

Proof. See [5, page 182]. □

Theorem 10. *Let $A \in \mathbf{C}^{n\times m}$ be a hub-dominant matrix with its columns denoted by $a_1,\ldots,a_m$ with $0 < \|a_1\|_2 \le \cdots \le \|a_{m-1}\|_2 \le \|a_m\|_2$. Let $S = A^H A \in \mathbf{C}^{m\times m}$ be the corresponding arrowhead dominant matrix with its eigenvalues denoted by $\lambda_1,\ldots,\lambda_m$ with $\lambda_1 \le \cdots \le \lambda_m$. Let $d^{(i)}$ and $\sigma^{(i)}$ denote the diagonal entry and the sum of magnitudes of off-diagonal entries, respectively, in row i of S for $i = 1,\ldots,m$. Then*

(a) $\mathrm{HubGap}_1(A)/2 \le \mathrm{EigenGap}_1(S)$, *and*
(b) $\mathrm{EigenGap}_1(S) = \lambda_m/\lambda_{m-1} \le (d^{(m-1)} + b + \sum_{i=1}^{m-2}\sigma^{(i)})/(d^{(m-2)} - \sigma^{(m-2)})$.

Proof. Let T be the matrix formed from S by deleting its last row and column. This means that T is a diagonally dominant matrix. Let the eigenvalues of T be $\{\mu_i\}$ with $\mu_1 \le \cdots \le \mu_{m-1}$. Then by Theorem 9, we have $\lambda_1 \le \mu_1 \le \lambda_2 \le \cdots \le \lambda_{m-1} \le \mu_{m-1} \le \lambda_m$. Applying Gershgorin's theorem to T and noting that T is a diagonally dominant with $d^{(m-1)}$ being its largest diagonal entry, we have $\mu_{m-1} \le 2d^{(m-1)}$. Thus $\lambda_{m-1} \le 2d^{(m-1)} = 2\|a_{m-1}\|_2^2$. As observed in the proof of Theorem 4, $\lambda_m \ge b = \|a_m\|_2^2$. Therefore, $\|a_m\|_2^2/(2\|a_{m-1}\|_2^2) \le \lambda_m/\lambda_{m-1}$ or $\mathrm{HubGap}_1(A)/2 \le \mathrm{EigenGap}_1(S)$.

Let E be the matrix formed from T with its diagonal entries replaced by the corresponding off-diagonal row sums, and let $\overline{T} = T - E$. Since T is a diagonally dominant matrix, $\overline{T}$ is a diagonal matrix with nonnegative diagonal entries. Let the diagonal entries of $\overline{T}$ be $\{\overline{d}^{(i)}\}$. Then $\overline{d}^{(i)} = d^{(i)} - \sigma^{(i)}$. Assume that $\overline{d}^{(1)} \le \cdots \le \overline{d}^{(m-1)}$. Since E is a symmetric diagonally dominant matrix with positive diagonal entries, it is a positive semidefinite matrix. Since $T = \overline{T}+E$, by Theorem 9 we have $\mu_i \ge \overline{d}^{(i)}$ for $i = 1,\ldots,m-1$. Let

$$S = \begin{pmatrix} D & c \\ c^H & b \end{pmatrix} \tag{38}$$

in accordance with Definition 9. By Theorem 3, we have $\sum_{i=1}^{m-1} d^{(i)} + b = \sum_{i=1}^{m}\lambda_m$. Thus, by noting $\lambda_1 \le \mu_1 \le \lambda_2 \le \cdots \le \lambda_{m-1} \le \mu_{m-1} \le \lambda_m$, we have

$$\begin{aligned} & d^{(1)} + d^{(2)} + \cdots + d^{(m-1)} + b \\ &\quad = \lambda_1 + \lambda_2 + \cdots + \lambda_m \ge \lambda_1 + \mu_1 + \cdots + \mu_{m-2} + \lambda_m \\ &\quad \ge \lambda_1 + \overline{d}^{(1)} + \cdots + \overline{d}^{(m-2)} + \lambda_m. \end{aligned} \tag{39}$$

This implies that $d^{(m-1)} + b + \sum_{i=1}^{m-2}\sigma^{(i)} \ge \lambda_1 + \lambda_m \ge \lambda_m$. Since $d^{(m-2)} - \sigma^{(m-2)} = \overline{d}^{(m-2)} \le \mu_{m-2} \le \lambda_{m-1}$, we have

$$\mathrm{EigenGap}_1(S) = \frac{\lambda_m}{\lambda_{m-1}} \le \frac{d^{(m-1)} + b + \sum_{i=1}^{m-2}\sigma^{(i)}}{d^{(m-2)} - \sigma^{(m-2)}}. \tag{40}$$

□

Note that if there exist positive numbers p and q, with $q < 1$, such that $(1-q)d^{(m-2)} \ge \sigma^{(m-2)}$ and

$$p\left(d^{(m-1)} + b\right) \ge \sum_{i=1}^{m-2}\sigma^{(i)}, \tag{41}$$

then the inequality (b) in Theorem 10 implies

$$\frac{\lambda_m}{\lambda_{m-1}} \le r\cdot\frac{d^{(m-1)} + b}{d^{(m-2)}}, \tag{42}$$

where $r = (1+p)/q$. As in the end of the proof of Theorem 4, it follows that

$$\mathrm{EigenGap}_1(S) \le r\cdot\left(\mathrm{HubGap}_1(A) + 1\right)\cdot\mathrm{HubGap}_2(A). \tag{43}$$

This together with (a) in Theorem 10 gives the following result.

Corollary 3. *Let $A \in \mathbf{C}^{n\times m}$ be a matrix with its columns $a_1,\ldots,a_m$ satisfying $0 < \|a_1\|_2^2 \le \cdots \le \|a_{m-1}\|_2^2 \le \|a_m\|_2^2$. Let $S = A^H A \in \mathbf{C}^{m\times m}$ be a Hermitian matrix with its eigenvalues $\lambda_1,\ldots,\lambda_m$ satisfying $0 \le \lambda_1 \le \cdots \le \lambda_m$. The following holds*

(1) if A is a hub-dominant matrix with $\|a_m\|_2 \gg \|a_{m-1}\|_2$, then S is an arrowhead dominant matrix with $\lambda_m \gg \lambda_{m-1}$; and
(2) if S is an arrowhead dominant matrix with $\lambda_m \gg \lambda_{m-1}$, and if $p(d^{(m-1)} + b) \ge \sum_{i=1}^{m-2}\sigma^{(i)}$ and $(1-q)d^{(m-2)} \ge \sigma^{(m-2)}$ for some positive numbers p and q with $q < 1$, then A is a hub-dominant matrix with $\|a_m\|_2 \gg \|a_{m-1}\|_2$ or $\|a_{m-1}\|_2 \gg \|a_{m-2}\|_2$ or both.

Sometimes, especially for large-dimensional matrices, it is desirable to relax the notion of diagonal dominance. This can be done using arguments analogous to those given above (see, e.g., [14]), and extensions represent an open research problem for the future.

5. CONCLUDING REMARKS

This paper has presented a hub matrix theory and applied it to beamforming MIMO communications systems. The fact that the performance of the MIMO beamforming scheme is critically related to the gap between the two largest eigenvalues of the channel propagation matrix is well known, but this paper reported for the first time how to obtain this insight directly from the structure of the matrix, that is, its hub properties. We believe that numerous communications systems might be well described within the formalism of hub matrices. As an example, one can consider the problem of noncooperative beamforming in a wireless sensor network, where several source (transmitting) nodes communicate with a destination node, but only one source node is located in the vicinity of the destination node and presents a direct line-of-sight to the destination node. Extending the hub matrix formalism to other types of matrices (e.g., matrices with a cluster of dominant columns) represents an interesting open research problem. The contributions reported in this paper can be extended further to treat the more general class of block arrowhead and hub dominant matrices that enable the analysis and design of algorithms and protocols in areas such as distributed beamforming and power control in wireless adhoc networks. By relaxing the diagonal-matrix condition, in the definition of an arrowhead matrix, with a block diagonal condition, and enabling groups of columns to be correlated or uncorrelated (orthogonal/nonorthogonal) in the definition of block dominant hub matrices, a much larger spectrum of applications could be treated within the proposed framework.

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