Completeness and Robustness Properties of Min-Wise Independent Permutations

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Abstract. We provide several new results related to the concept of min-wise independence. Our main result is that any randomized sampling scheme for the relative intersection of sets based on testing equality of samples yields an equivalent min-wise independent family. Thus, in a certain sense, min-wise independent families are "complete" for this type of estimation.

We also discuss the notion of robustness, a concept extending min-wise independence to allow more efficient use of it in practice. A surprising result arising from our consideration of robustness is that under a random permutation from a min-wise independent family, any element of a fixed set has an equal chance to get any rank in the image of the set, not only the minimum as required by definition.

1 Introduction

A family of permutations $\mathcal{P} \subseteq S_n$ is called *min-wise independent* (abbreviated MWI) if for any set $X \subseteq [n] = \{1, \ldots, n\}$ and any $x \in X$, when π is chosen at random in $\mathcal{P}$ according to some specified probability distribution we have

$$\mathbf{Pr}\big(\min\{\pi(X)\} = \pi(x)\big) = \frac{1}{|X|} \quad . \tag{1}$$

In other words we require that all the elements of any fixed set X have an equal chance to become the minimum element of the image of X under π.

When the distribution on $\mathcal{P}$ is non-uniform, the family is called *biased*, and it is called *unbiased* otherwise. In general in this paper we will not specify the probability distribution on $\mathcal{P}$ unless relevant, and from now on when we say "π chosen at random in (the min-wise independent family) $\mathcal{P}$" we mean "π chosen in $\mathcal{P}$ according to the probability distribution associated to $\mathcal{P}$ such that (1) holds."

Together with Moses Charikar and Alan Frieze, we introduced this notion in [4] motivated by the fact that such a family (under some relaxations) is essential

* Parts of this work were done while this author was at Compaq Systems Research Center.

to the algorithm used in practice by the AltaVista web index software to detect and filter near-duplicate documents. The crucial property that enables this application is the following: let X be a subset of $[n]$. Pick a "sample" $s(X) \in X$ by choosing at random a permutation π from a family of permutations $\mathcal{P}$ and letting

$$s(X) = \pi^{-1}(\min\{\pi(X)\}) \ . \tag{2}$$

Then, if $\mathcal{P}$ is a MWI-family, for any two nonempty subsets A and B, we have

$$\mathbf{Pr}\big(s(A) = s(B)\big) = \frac{|A \cap B|}{|A \cup B|} \ . \tag{3}$$

Hence such samples can be used to estimate the relative size of the intersection of sets, a quantity that we call the *resemblance* of A and B, defined as

$$R(A, B) = \frac{|A \cap B|}{|A \cup B|} \ . \tag{4}$$

We estimate resemblance by first picking, say, 100 permutations from a MWI-family, and then computing samples for each set of interest. Then the resemblance of any two sets can be estimated simply by determining the fraction of samples that coincide.

In practice we can allow small relative errors. We say that $\mathcal{P} \subseteq S_n$ is *approximately min-wise independent with relative error* ϵ (or just approximately min-wise independent, where the meaning is clear) if for any set $X \subseteq [n]$ and any $x \in X$, when π is chosen at random in $\mathcal{P}$ we have

$$\left| \mathbf{Pr}\big(\min\{\pi(X)\} = \pi(x)\big) - \frac{1}{|X|} \right| \leq \frac{\epsilon}{|X|} \ . \tag{5}$$

For further details about the use of these ideas to estimate document similarity see [6, 1, 2]. An optimal (size-wise) construction for a MWI-family was obtained by Takei, Itoh, and Shinozaki [13]. Explicit constructions of approximately MWI-families were obtained by Indyk [8] and by Saks & al. [11]. For an application of these families to derandomization see [5].

We also note that concepts similar to min-wise independence have appeared prior to our work [4] as well. For example, the monotone ranged hash functions described in [9] have the min-wise independence property; Cohen [7] uses the property that the minimum element of a random permutation is uniform to estimate the size of the transitive closure, as well as to solve similar related problems; and Mulmuley [10] uses what we call approximate min-wise independence to use fewer random bits for several randomized geometric algorithms.

The main result of this paper, presented in Sect. 2, is that, rather surprisingly, *any* sampling scheme that has property (3) is equivalent to a scheme derived via equation (2) from a min-wise independent family of permutations. More precisely we have the following theorem:

Theorem 1. *Let $\mathcal{F}$ be a family of functions from nonempty subsets of $[n]$ to some arbitrary set Ω. Assume there exists a probability distribution on $\mathcal{F}$ such that for any two nonempty subsets, A and B,*

$$\mathbf{Pr}\big(f(A) = f(B)\big) = \frac{|A \cap B|}{|A \cup B|} \ .$$

Then there exists a min-wise independent family of permutations $\mathcal{P}$ such that every $f \in \mathcal{F}$ is defined by

$$f(X) = f\left(\left\{\pi_f^{-1}(\min\{\pi_f(X)\})\right\}\right)$$

for some $\pi_f \in \mathcal{P}$.

We note here some immediate consequences of the theorem:

(a) The induced family of permutations has the same size as the initial family of functions, that is $|\mathcal{P}| = |\mathcal{F}|$.
(b) Each $f \in \mathcal{F}$ takes exactly n distinct values $f(\{x_1\}), \ldots, f(\{x_n\})$. (A priori each f can take $2^n - 1$ values.)
(c) Assume that we add the condition that for every $X \subseteq [n]$, each $f \in \mathcal{F}$ satisfies $f(X) \in X$; in other words, the "sample" must belong to the set being sampled. Then for every $x \in [n]$ each f satisfies $f(\{x\}) = x$, and hence each f has the form

$$f(X) = \pi_f^{-1}(\min\{\pi_f(X)\}) \ .$$

(The converse of the assumption is also true: if for every $x \in [n]$ we have $f(\{x\}) = x$ then $f(X) \in X$ follows. See Corollary 1 below.)
(d) Thus every estimation scheme that has property (3) is equivalent under renaming to a sampling scheme derived via equation (2) from a min-wise independent family of permutations. (For each f, $f(\{x_1\})$ is the "name" of x_1, $f(\{x_2\})$ is the "name" of x_2, etc.)

Of course in practice it might be more convenient to represent $\mathcal{F}$ directly rather than via $\mathcal{P}$. (See [3] for an example.) But the fact remains that any method of sampling to estimate resemblance via equation (3) is equivalent to sampling with min-wise independent permutations.

To develop some intuition, before plunging into the proof, we start by observing that the choice of "min" in the definition (1) is somewhat arbitrary. Clearly if we replace "min" with "max" both in (1) and in (2), property (3) holds. More generally, we can fix a permutation $\sigma \in S_n$ (think of it as a total order on $[n]$), and require $\mathcal{P}$ to satisfy the property

$$\mathbf{Pr}\big(\min\{\sigma\big(\pi(X)\big)\} = \sigma\big(\pi(x)\big)\big) = \frac{1}{|X|} \ . \tag{6}$$

Then we can choose samples according to the rule

$$s(X) = \pi^{-1}\left(\sigma^{-1}\left(\min\{\sigma(\pi(X))\}\right)\right) \ .$$

(We obtain "max" by taking $\sigma(i) = n + 1 - i$.)

Is there any advantage to choosing a particular σ? A moment of reflection indicates that there is nothing to be gained since we can simply replace the family $\mathcal{P}$ by the family $\mathcal{P} \circ \sigma$. This is, in fact, a very simple instance of Theorem 1. However, it could be of interest if a family $\mathcal{P}$ satisfies condition (6) with respect to more than one order σ. One reason is that, in practice, computing $\pi(X)$ is expensive (see [3] for details). If a family has the min-wise independence property with respect to several orders, then we can extract a sample for each order. Obviously these samples are correlated, but if the correlation can be bounded, these samples are still usable.

Takei, Itoh, and Shinozaki [13] presented an optimal (size-wise) construction for a MWI-family under the uniform distribution. Their family has size $\mathrm{lcm}(1, \ldots, n)$, matching the lower bound of [4]. They observed that their construction produces a family that is simultaneously min-wise independent and max-wise independent. In Sect. 3 we show that this is not a fluke; in fact, any min-wise independent family is also max-wise independent. Moreover, if $\mathcal{P} \subseteq S_n$ is min-wise independent, then for any set $X \subseteq [n]$, any $x \in X$, and any fixed $r \in \{1, \ldots, |X|\}$, when π is chosen at random in $\mathcal{P}$ we have

$$\mathbf{Pr}\big(\mathrm{rank}(\pi(x), \pi(X)) = r\big) = \frac{1}{|X|} \ , \tag{7}$$

where $\mathrm{rank}(x, X)$ for $x \in X$ is the number of elements in X not greater than x. Hence the max-wise independence property follows by taking $r = |X|$.

In Sect. 4 we discuss families that have the min-wise independence property with respect to *all* possible orders σ. We call such families *robust*. We show that although not every min-wise independent family is robust, there are non-trivial robust families. On the other hand, robust families under the uniform distribution of size $\mathrm{lcm}(1, \ldots, n)$ do not necessarily exist for every n.

2 Any Sampling Scheme is a MWI-Family

In this section we prove the following:

Theorem 1 *Let $\mathcal{F}$ be a family of functions from nonempty subsets of $[n]$ to some arbitrary set Ω. Assume there exists a probability distribution on $\mathcal{F}$ such that for any two nonempty subsets, A and B,*

$$\mathbf{Pr}\big(f(A) = f(B)\big) = \frac{|A \cap B|}{|A \cup B|} \ .$$

Then there exists a min-wise independent family of permutations $\mathcal{P}$ such that every $f \in \mathcal{F}$ is defined by

$$f(X) = f\left(\left\{\pi_f^{-1}(\min\{\pi_f(X)\})\right\}\right)$$

for some $\pi_f \in \mathcal{P}$.

Proof. Assume the premises of the Theorem. We start with two Lemmas.

Lemma 1. *Let X be a nonempty subset of $[n]$. Then for any $x \in X$*

$$\mathbf{Pr}(f(X) = f(\{x\})) = \frac{|X \cap \{x\}|}{|X \cup \{x\}|} = \frac{1}{|X|} \ .$$

Corollary 1. *Let $X = \{x_1, x_2, \ldots, x_k\}$ be a nonempty subset of $[n]$. Then for each $f \in \mathcal{F}$*

$$f(X) \in \{f(\{x_1\}), f(\{x_2\}), \ldots, f(\{x_k\})\} \ .$$

Proof.

$$\begin{aligned} &\mathbf{Pr}\left(f(X) \in \{f(\{x_1\}), f(\{x_2\}), \ldots, f(\{x_k\})\}\right) \\ &\qquad = \sum_{i=1}^{k} \mathbf{Pr}\left(f(X) = f(\{x_i\})\right) = 1. \end{aligned}$$

□

Lemma 2. *Let $X = \{x_1, x_2, \ldots, x_k\}$ and Y be a nonempty subsets of $[n]$. If $X \subseteq Y$, then for every $f \in \mathcal{F}$, if $f(Y) \in \{f(\{x_1\}), f(\{x_2\}), \ldots, f(\{x_k\})\}$, then $f(Y) = f(X)$.*

Proof. By hypothesis

$$\mathbf{Pr}(f(X) = f(Y)) = \frac{|X \cap Y|}{|X \cup Y|} = \frac{k}{|Y|} \ .$$

On the other hand,

$$\begin{aligned} &\mathbf{Pr}(f(X) = f(Y)) \\ &\quad = \mathbf{Pr}(f(X) = f(\{x_1\}) \wedge f(Y) = f(\{x_1\})) + \cdots \\ &\qquad \cdots + \mathbf{Pr}(f(X) = f(\{x_k\}) \wedge f(Y) = f(\{x_k\})) \\ &\quad = \mathbf{Pr}(f(X) = f(\{x_1\}) \mid f(Y) = f(\{x_1\}))\, \mathbf{Pr}(f(Y) = f(\{x_1\})) + \cdots \\ &\qquad \cdots + \mathbf{Pr}(f(X) = f(\{x_k\}) \mid f(Y) = f(\{x_k\}))\, \mathbf{Pr}(f(Y) = f(\{x_k\})) \\ &\quad = \mathbf{Pr}(f(X) = f(\{x_1\}) \mid f(Y) = f(\{x_1\}))(1/|Y|) + \cdots \\ &\qquad \cdots + \mathbf{Pr}(f(X) = f(\{x_k\}) \mid f(Y) = f(\{x_k\}))(1/|Y|). \end{aligned}$$

(The last equality follows from Lemma 1.) Hence for every $x_i \in X$

$$\mathbf{Pr}(f(X) = f(\{x_i\}) | f(Y) = f(\{x_i\})) = 1 \ ,$$

and therefore for every $f \in F$, if $f(Y) = f(\{x_i\})$ then $f(X) = f(\{x_i\})$ as well.

□

Lemma 3. *For any two distinct elements $x_1, x_2 \in [n]$ and each $f \in \mathcal{F}$.*

$$f(\{x_1\}) \neq f(\{x_2\}) \ .$$

Proof. By hypothesis $\mathbf{Pr}(f(\{x_1\}) = f(\{x_2\})) = 0$. $\square$

Returning to the proof of the Theorem, we show now how to construct for each $f \in \mathcal{F}$ a permutation π_f such that for every nonempty set X

$$f(X) = f\left(\left\{\pi_f^{-1}(\min\{\pi_f(X)\})\right\}\right) \quad . \tag{8}$$

Note that the family $\mathcal{P}$ given by the π_f above are clearly min-wise independent by Lemma 1.

Fix f and let $g : \{f(\{x_1\}), \ldots, f(\{x_n\})\} \to [n]$ be the function defined by $g(f(\{x_i\})) = x_i$. In view of Lemma 3 g is well-defined. Now define a sequence $y_1, y_2, \ldots, y_n$ as follows:

$$\begin{aligned}
y_1 &= g(f([n])) \\
y_2 &= g(f([n] \setminus \{y_1\})) \\
y_3 &= g(f([n] \setminus \{y_1, y_2\})) \\
&\vdots
\end{aligned}$$

In view of Corollary 1 g is correctly used and we have

$$\begin{aligned}
f([n]) &= f(\{y_1\}) \\
f([n] \setminus \{y_1\})) &= f(\{y_2\}) \\
f([n] \setminus \{y_1, y_2\})) &= f(\{y_3\}) \\
&\vdots
\end{aligned}$$

Furthermore $y_1, y_2, \ldots, y_n$ is a permutation of $[n]$. Finally we take π_f to be the inverse of the permutation determined by the y_i; that is, π_f maps y_1 to 1, y_2 to 2, etc. We need to show that f satisfies equation (8) for every nonempty set X.

Fix X and consider the sets $Y_1 = [n]$, $Y_2 = [n] \setminus \{y_1\}$, $Y_3 = [n] \setminus \{y_1, y_2\}$, ..., $Y_n = \{y_n\}$. Let k be the largest index such that Y_k still includes X. This implies that

(a) $y_k \in X$ since otherwise we could have taken Y_{k+1}.
(b) $\{y_1, y_2, \ldots, y_{k-1}\} \cap X = \emptyset$ since none of these elements belong to Y_k.

By definition $f(Y_k) = f(\{y_k\})$. But $y_k \in X \subseteq Y_k$ and therefore Lemma 2 implies that $f(X) = f(\{y_k\})$ as well. On the other hand property (a) above implies that $\min\{\pi_f(X)\} \leq k$ and property (b) implies that $\min\{\pi_f(X)\} > k - 1$. Hence $\min\{\pi_f(X)\} = k$ and $\pi_f^{-1}(\min\{\pi_f(X)\}) = y_k$ as required. $\square$

3 Rank Uniformity for *MWI*-Families

In this section, we show that any min-wise independent family actually has the property that every item in any fixed set is equally likely to have any rank in the image of the set – not just the minimum rank as required by definition. Our analysis is based on the following lemma, proven in [12]. (Alternatively, the "only if" part follows also from Theorem 6 of [4] and the "if" part follows from the proof of Theorem 2 below.)

Lemma 4. *A family of permutations $\mathcal{P}$ is min-wise independent if and only if for any set $X \subset [n]$ of size k and any element $x \in [n] \setminus X$*

$$\mathbf{Pr}\left(\pi(X) = [k] \wedge \pi(x) = k+1\right) = \frac{1}{\binom{n}{k}(n-k)},$$

when π is chosen at random in $\mathcal{P}$.

In other words, if we fix a set X of size k and an extra element x, the probability that x maps to $k+1$ and X maps to $\{1, \ldots, k\}$ in some arbitrary order is exactly what "it should be" if we were sampling uniformly from the entire set of permutations S_n.

Theorem 2. *If $\mathcal{P}$ is min-wise independent, and π is chosen at random from $\mathcal{P}$, then*

$$\mathbf{Pr}\big(\mathrm{rank}(\pi(x), \pi(X)) = r\big) = \frac{1}{|X|} \ . \tag{9}$$

Proof. We sum over all the possible ways such that $\mathrm{rank}(\pi(x), \pi(X)) = r$ and $\pi(x) = s$ and consider which elements map to $[s-1]$. Note that we must have $r \leq s \leq n - (|X| - r)$. There must be $r - 1$ other elements of X, call them $\{x_1, x_2, \ldots, x_{r-1}\}$, such that $\pi(x_i) \in [s-1]$, and there are $\binom{|X|}{r-1}$ ways to choose them. Similarly, there must be $s - r$ elements of $[n] \setminus X$, call them $\{y_1, y_2, \ldots, y_{n-r}\}$, such that $\pi(y_i) \in [s-1]$ and there are $\binom{n-|X|}{s-r}$ ways to choose these elements. For each possible combination of choices, we have from Lemma 4 that the probability that these elements are mapped to $[s-1]$ and x is mapped to s is

$$\frac{1}{\binom{n}{s-1}(n-s+1)} \ .$$

Hence

$$\begin{aligned}
\mathbf{Pr}\big(\mathrm{rank}(\pi(x), \pi(X)) = r\big) &= \sum_{s=r}^{n-|X|+r} \frac{\binom{|X|-1}{r-1}\binom{n-|X|}{s-r}}{\binom{n}{s-1}(n-s+1)} \\
&= \frac{1}{|X|\binom{n}{|X|}} \sum_{s=r}^{n-|X|+r} \binom{s-1}{r-1}\binom{n-s}{|X|-r} \\
&= \frac{1}{|X|\binom{n}{|X|}} \binom{n}{|X|} = \frac{1}{|X|} \ .
\end{aligned}$$

(The second equality is obtained by expanding binomials into factorials and regrouping. The third equality is obtained by counting the ways of choosing $|X|$ elements out of $[n]$ by summing over all possible values s for the r'th largest element among those chosen.) $\square$

4 Robust Families

We now consider *robust* families. As described in the introduction, robustness is an extension of min-wise independence. Formally, a family $\mathcal{P}$ is robust if for *every* possible permutation σ, when π is chosen at random in $\mathcal{P}$

$$\mathbf{Pr}\big(\min\{\sigma\big(\pi(X)\big)\} = \sigma\big(\pi(x)\big)\big) = \frac{1}{|X|} \quad . \tag{10}$$

Trivially, S_n is a robust family. We first demonstrate that there exist non-trivial robust families. To this end, we extend the condition for min-wise independent families given in Lemma 4 to the equivalent condition for robust families. Since robust families are min-wise independent under any order σ we obtain the following:

Lemma 5. *A family of permutations $\mathcal{P}$ is robust if and only if for any set $X \subset [n]$ of size k and any element $x \in [n] \setminus X$, and any other set $A \subset [n]$ of size also k and any element $a \in [n] \setminus A$*

$$\mathbf{Pr}\big(\pi(X) = A \wedge \pi(x) = a\big) = \frac{1}{\binom{n}{k}(n-k)} \quad . \tag{11}$$

Theorem 3. *There exist biased robust families of size at most*

$$n^2 \binom{2(n-1)}{n-1} \quad .$$

Proof. Following an idea used in [4], we establish a linear program for determining a robust family of the required size. There are $n!$ variables x_{π_i}, one for each possible permutation π_i. The variable x_{π_i} represent the probability that π_i is chosen within our family; if $x_{\pi_i} = 0$, we may exclude π_i from the family.

Our linear program is based on Lemma 5. We set up an equation for each pair (a, A) and (x, X) with $|A| = |X|$, with each equation representing the constraint that (a, A) maps to (x, X) with the required probability. Hence there are

$$\sum_{i=0}^{n-1} n^2 \binom{n-1}{i}^2 = n^2 \binom{2(n-1)}{n-1}$$

equations. We know there exists a solution to the linear program, since if each permutation is chosen with probability $1/n!$ we have a robust family. Hence there must be a basic feasible solution with at most $n^2\binom{2(n-1)}{n-1}$ variables taking non-zero values. This solution yields a biased robust family. □

It is also worthwhile to ask if there are any non-trivial unbiased robust families. We demonstrate that in fact there are non-trivial families for $n \geq 4$.

Recall that the permutations S_n can be split into two groups, each of size $n!/2$, as follows: a permutation is called *even* if it can be obtained by an even number of transpositions from the identity, and odd *odd* otherwise.

Theorem 4. *For $n \geq 4$, the even permutations and the odd permutations of* $[n]$ *both yield robust families.*

Proof. We use Lemma 5. That is, we must show that for each pair (x, X) with $x \in [n]$, $X \subseteq [n]$, $x \notin X$, the probability that $\pi(x) = a$ and $\pi(X) = A$ is correct for every (a, A) with $a \in [n]$, $A \subseteq [n]$, $|A| = |X|$, and $a \notin A$.

Equivalently, since the odd permutations and even permutations divide the set of all permutations into two equal-sized families, it suffices to show that the number of even permutations mapping (x, X) into (a, A) is the same as the number of odd permutations that do so. Note that as $n \geq 4$, either $|X| \geq 2$ or $|[n] - X - \{x\}| \geq 2$. In the first case, we can determine a one-to-one mapping of even permutations to odd permutations that map (x, X) into (a, A) by choosing two particular elements of X (say the two smallest) and transposing them. In the second case, we may do the same by transposing two elements of $[n] - X - \{x\}$. □

From the lower bound in [4], we know that unbiased min-wise independent families (and hence robust families) have size at least $\text{lcm}(1, \ldots, n)$. As $\text{lcm}(1, \ldots, n) = n!/2$ for $n = 4$ and $n = 5$, the result of Theorem 4 is optimal for these cases. We suspect that Theorem 4 is in fact optimal for all $n \geq 4$; that is, there is no unbiased robust family of size less than $n!/2$. While we cannot yet show this, we can show that for $n = 6$, there is no unbiased robust family of size $\text{lcm}(1, \ldots, n) = 60$.

Theorem 5. *All the unbiased robust families of permutations of* $\{1, 2, 3, 4, 5, 6\}$ *have size greater than 60.*

Proof. The proof uses an exhaustive search, where the search for a robust family is reduced by using symmetry and Lemma 5. Details will appear in the full paper. □

Given the development of approximate min-wise independent families of permutations developed in [4], it is natural to ask about approximate robust families of permutations as well. A family of permutations is said to be *approximately robust with relative error* ϵ if and only if for every permutation order σ,

$$\left| \mathbf{Pr}\big(\min\{\sigma\big(\pi(X)\big)\} = \sigma\big(\pi(x)\big)\big) - \frac{1}{|X|} \right| \leq \frac{\epsilon}{|X|} \ . \tag{12}$$

That is, regardless of σ, the probability over the choice of π that an element x is the minimum of a set $|X|$ is within a factor of $(1 \pm \epsilon)$ of the natural probability $\frac{1}{|X|}$. It is straightforward to show that there must be small approximate robust families.

Theorem 6. *There are approximate robust families of size* $O(n^2 \log(n)/\epsilon)$.

Proof. The proof follows Theorem 3 of [4]. We simply choose a random set of permutations of the appropriate size, and show that with some probability, we obtain an unbiased approximate robust family. Details will appear in the full paper. □

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